

MASTER'S THESIS

Development of a
Quantum-Teleportation-Based Noise
Reduction Technique for
Gravitational-Wave Detectors

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Abstract

Since the first direct detection of gravitational waves in 2015, gravitational-wave astronomy has rapidly developed as a new field. It now provides a powerful tool for testing general relativity in the strong-field regime and for exploring previously inaccessible phenomena in the universe such as black holes and the early universe. At present, the next-generation gravitational-wave detector, the Einstein Telescope (ET), is being planned in Europe. Once in operation, ET is expected to detect more than 500 times as many high signal-to-noise ratio events as current detectors, enabling a dramatic advancement in astrophysics. The sensitivity of gravitational-wave detectors is limited over a broad frequency band by quantum noise arising from quantum fluctuations of the optical field entering the detector. One of the most promising techniques for reducing quantum noise is the injection of frequency-dependent squeezed states. In the currently operating LIGO detectors, sensitivity improvement has been achieved by injecting frequency-dependent squeezing using a 300 m-long filter cavity. In contrast, ET is designed as a detuned interferometer that includes an optical cavity detuned from resonance inside the interferometer to enhance sensitivity in a specific frequency band. As a result, implementing frequency-dependent squeezing in ET requires two kilometer-scale filter cavities to be installed underground, posing significant technical and economic challenges. Motivated by these challenges, a novel scheme has recently been proposed in which frequency-dependent squeezing optimized for a detuned interferometer is injected using quantum teleportation. This approach enables the realization of frequency-dependent squeezing while avoiding the technical difficulties and optical losses associated with long filter cavities.

In this study, we aim to experimentally demonstrate this scheme by designing and implementing the required experimental setup. We first introduce and characterize two types of optical cavities required for the experiment. These include a frequency-separating cavity and test cavities designed to reproduce the frequency-dependent response of a detuned interferometer. Both types of cavities are actively controlled, and their optical properties are experimentally evaluated. We then implement a balanced homodyne detector and optimize its operating conditions. Using this detector, we observe squeezing of approximately 1.4 dB below the shot-noise level, generated via optical parametric oscillation. By incorporating optical losses into a theoretical model based on experimentally measured parameters, we estimate that a quantum noise reduction of approximately 0.5 dB could be achieved in the present experimental system.

This work represents an important experimental step toward the demonstration of a quantum-teleportation-based scheme for frequency-dependent squeezing without the use of filter cavities.

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Chapter 1

Introduction

Gravitational waves are ripples in spacetime predicted by general relativity that propagate at the speed of light. In 1974, a decrease in the orbital period of a binary pulsar system known as PSR B1913+16 was reported [3]. It was later shown that the observed orbital decay agrees with high precision with the decrease predicted by general relativity due to energy loss through gravitational radiation [4, 5]. This result provided the first indirect evidence for the existence of gravitational waves. In 2015, the Laser Interferometer Gravitational-Wave Observatory (LIGO) in the United States achieved the first direct detection of gravitational waves from a binary black hole merger with component masses of $36 M_{\odot}$ and $24 M_{\odot}$, an event designated GW150914 [6]. In 2017, gravitational waves from a binary neutron star merger, GW170817, were detected by a network of three detectors, LIGO and Advanced Virgo [7]. Approximately 1.7 seconds after the merger, a short gamma-ray burst, GRB 170817A, was observed by the *Fermi* Gamma-ray Space Telescope [8]. This observation provided direct evidence supporting the hypothesis that short gamma-ray bursts originate from neutron star mergers and marked the beginning of multi-messenger astronomy.

A major turning point in the history of gravitational-wave detection dates back to the 1960s. At the University of Maryland, Weber proposed resonant-mass detectors designed to measure gravitational-wave tidal forces as small changes in the resonant modes of aluminum cylinders. Although Weber reported the detection of gravitational waves in 1968 [9], these results were not generally accepted. Subsequently, in the 1970s, the concept of laser-interferometric gravitational-wave detectors was proposed. In the 1990s, large-scale interferometric detectors such as LIGO (USA), VIRGO (Italy), GEO600 (Germany), and TAMA300 (Japan) were constructed. These instruments are referred to as first-generation gravitational-wave detectors. In the 2010s, second-generation detectors—Advanced LIGO, Advanced Virgo, and KAGRA—were developed as upgrades to the first-generation instruments with the goal of achieving routine direct detections. The first observing run (O1) began in 2015, and by November 2025 the fourth observing run (O4) had been completed. The cumulative number of detected events is shown in Fig. 1.1. While approximately 90 events were detected during O1 through O3, nearly three times as many events were observed during O4 alone. At present, commissioning activities are underway in preparation for the fifth observing run (O5), and plans are also advancing for third-generation detectors such as the Einstein Telescope, Cosmic Explorer, and the space-based detector LISA.

As detector sensitivities have continued to improve, quantum measurement limits, commonly referred to as the standard quantum limit—have become an increasingly significant

issue. During O3, frequency-independent squeezing was implemented by applying phase squeezing to the vacuum fluctuations entering the interferometer, resulting in a reduction of shot noise. However, this improvement comes at the cost of increased uncertainty in the amplitude quadrature, leading to enhanced radiation pressure noise. In the LIGO A+ upgrade deployed during O4, further improvements were achieved through enhancements to the laser system and increased circulating power, along with the installation of a 300-m filter cavity [10]. By realizing frequency-dependent squeezing, quantum noise reductions of 5.2 dB at the Hanford site and 6.1 dB at Livingston were achieved [11]. These developments demonstrate that the implementation of quantum noise reduction techniques has become indispensable for current and future gravitational-wave detectors.

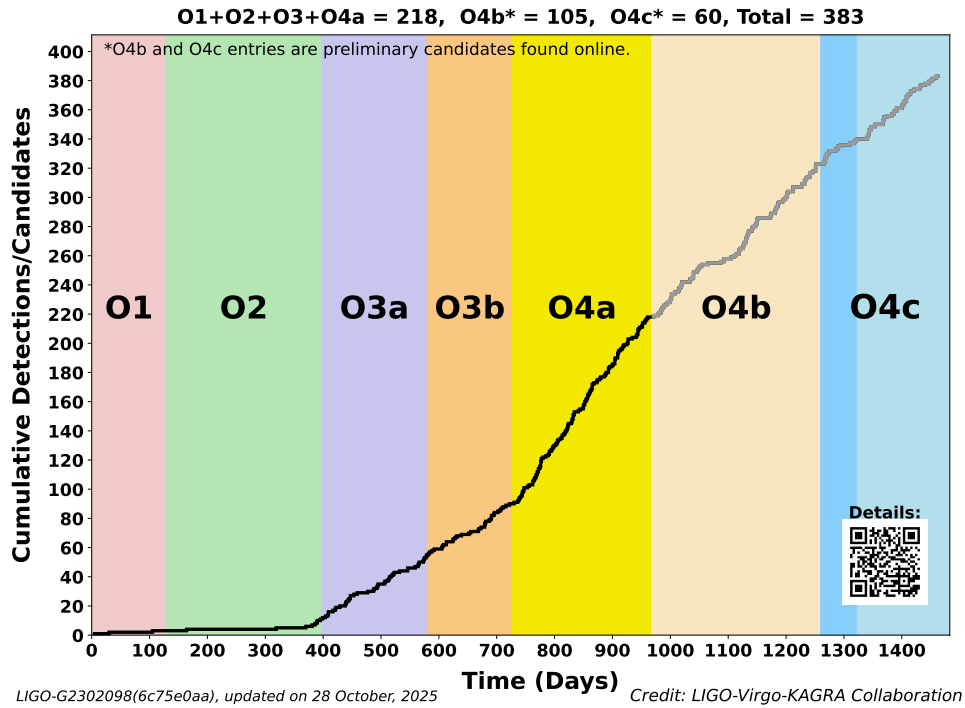


Figure 1.1: Cumulative number of detected events [1].

Chapter 2

Gravitational Wave

In this chapter, we describe the theoretical background of gravitational waves and the principles of their detection. In Section 2.1, gravitational waves are derived from the Einstein equations in the weak-field approximation, and possible astrophysical sources of gravitational waves are briefly introduced. In Section 2.2, the basic principles and configurations of laser-interferometric gravitational wave detectors are presented. In Section 2.3, the dominant noise sources that limit the detector sensitivity are discussed.

2.1 Derivation of gravitational waves

In this section, we review the theoretical framework of gravitational waves. In a spacetime described by a metric tensor $g_{\mu\nu}$, the infinitesimal spacetime interval ds between two neighboring points is given by

$$ds^2 = \sum_{\mu=0}^3 \sum_{\nu=0}^3 g_{\mu\nu} dx^\mu dx^\nu \quad (2.1.1)$$

$$:= g_{\mu\nu} dx^\mu dx^\nu,$$

where the second line uses the Einstein summation convention. Throughout this chapter, repeated indices within a single term are implicitly summed over, and Greek indices take the values 0, 1, 2, 3. The spacetime interval defined in Eq. (2.1.1) is a Lorentz-invariant scalar quantity determined by the metric tensor, which characterizes the geometry of spacetime. According to general relativity, the metric tensor satisfies the Einstein field equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (2.1.2)$$

where G is the gravitational constant, $G_{\mu\nu}$ is the Einstein tensor, and $T_{\mu\nu}$ is the energy-momentum tensor. The Einstein tensor is expressed in terms of the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \quad (2.1.3)$$

For completeness, we summarize the definitions of the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R , the Riemann tensor $R^\mu_{\nu\lambda\kappa}$, and the Christoffel symbols $\Gamma^\mu_{\nu\lambda}$ ¹:

¹Partial derivatives are denoted as $\partial_\nu A_\mu = A_{\mu,\nu}$.

$$\begin{aligned}
R_{\mu\nu} &= R_{\mu\alpha\nu}^{\alpha}, \quad R = R_{\mu}^{\mu}, \\
R_{\nu\lambda\kappa}^{\mu} &= \Gamma_{\nu\kappa,\lambda}^{\mu} - \Gamma_{\nu\lambda,\kappa}^{\mu} + \Gamma_{\sigma\lambda}^{\mu} \Gamma_{\nu\kappa}^{\sigma} - \Gamma_{\sigma\kappa}^{\mu} \Gamma_{\nu\lambda}^{\sigma}, \\
\Gamma_{\nu\lambda}^{\mu} &= \frac{1}{2} g^{\mu\sigma} (g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\nu\lambda,\sigma}).
\end{aligned} \tag{2.1.4}$$

A flat spacetime in the absence of gravitational fields is called Minkowski spacetime, whose metric tensor $\eta_{\mu\nu}$ is given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.1.5}$$

2.1.1 Einstein equations in the weak-field approximation

We assume that the gravitational field is sufficiently weak so that the metric tensor can be written as a linear perturbation of the Minkowski metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1. \tag{2.1.6}$$

In this subsection, we derive the Einstein equations under this perturbative assumption. Neglecting terms of second and higher order in the metric perturbation, the Christoffel symbols $\Gamma_{\nu\lambda}^{\mu}$ are given by

$$\Gamma_{\nu\lambda}^{\mu} = \frac{1}{2} \eta^{\mu\sigma} (h_{\sigma\nu,\lambda} + h_{\sigma\lambda,\nu} - h_{\nu\lambda,\sigma}), \tag{2.1.7}$$

where we have used the fact that $\partial_{\lambda} \eta_{\mu\nu} = 0$. Under the same approximation, the Ricci tensor and the Ricci scalar are expressed as

$$\begin{aligned}
R_{\mu\nu} &= \frac{1}{2} (h_{\beta\nu,\mu}^{\beta} - \square h_{\mu\nu} - h_{,\mu\nu} + h_{\mu\alpha,\nu}^{\alpha}), \\
R &\simeq \eta_{\mu\nu} R_{\mu\nu} \\
&= h_{\mu\nu}^{\mu\nu} - \square h,
\end{aligned} \tag{2.1.8}$$

where $\square = \eta^{\mu\nu} \partial_{\mu} \partial_{\nu}$ is the d'Alembert operator and $h = h_{\mu}^{\mu} = \eta^{\mu\nu} h_{\mu\nu}$. Therefore, the Einstein tensor $G_{\mu\nu}$ is written as

$$G_{\mu\nu} = \frac{1}{2} \left(h_{\beta\nu,\mu}^{\beta} - \square h_{\mu\nu} - h_{,\mu\nu} + h_{\mu\alpha,\nu}^{\alpha} - \eta_{\mu\nu} (h_{\alpha\beta}^{\alpha\beta} - \square h) \right). \tag{2.1.9}$$

Introducing the trace-reversed perturbation

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad \bar{h} = \bar{h}_{\mu}^{\mu} = -h, \tag{2.1.10}$$

the Einstein tensor can be rewritten as

$$G_{\mu\nu} = \frac{1}{2} \left(\bar{h}_{\beta\nu,\mu}{}^{\beta} - \square \bar{h}_{\mu\nu} + \bar{h}_{\mu\alpha,\nu}{}^{\alpha} - \eta_{\mu\nu} \bar{h}_{\alpha\beta}{}^{\alpha\beta} \right). \quad (2.1.11)$$

Since there are four degrees of freedom in the choice of coordinates, we impose the Lorentz gauge condition

$$\bar{h}^{\mu\nu}{}_{,\mu} = 0. \quad (2.1.12)$$

Under this condition, the Einstein equations reduce to

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}. \quad (2.1.13)$$

In vacuum, where $T_{\mu\nu} = 0$, the Einstein equations reduce to the wave equation

$$\square \bar{h}_{\mu\nu} = 0. \quad (2.1.14)$$

2.1.2 Propagation of gravitational waves

We solve Eq. (2.1.14), which was derived under the weak-field approximation, and obtain a monochromatic plane-wave solution of the form

$$\bar{h}_{\mu\nu} = a_{\mu\nu} e^{ik_{\alpha} x^{\alpha}}. \quad (2.1.15)$$

Here, $a_{\mu\nu}$ is referred to as the polarization tensor of the gravitational wave. Since $\bar{h}_{\mu\nu}$ is a symmetric tensor and satisfies the Lorentz gauge condition given in Eq. (2.1.12), the polarization tensor initially has $10 - 4 = 6$ independent degrees of freedom. However, the Lorentz gauge condition remains invariant under an infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} + \xi^{\mu}, \quad \square \xi^{\mu} = 0, \quad |\xi^{\mu}| \ll 1.$$

This residual gauge freedom allows us to further reduce the number of independent degrees of freedom by four. We impose the transverse-traceless (TT) gauge conditions:

$$\begin{aligned} \bar{h}_{\mu 0} &= 0 \quad (\text{purely spatial components}), \\ \bar{h}^{\mu\nu}{}_{,\mu} &= 0 \quad (\text{Lorentz gauge}), \\ \bar{h}^{\mu}{}_{\mu} &= 0 \quad (\text{traceless condition}). \end{aligned} \quad (2.1.16)$$

Under these conditions, the polarization tensor becomes a symmetric tensor with only $10 - 8 = 2$ independent degrees of freedom.

Next, we consider a plane gravitational wave propagating in the z -direction under the TT gauge:

$$h_{\mu\nu}^{\text{TT}} = a_{\mu\nu} \exp[-ik(ct - z)]. \quad (2.1.17)$$

From the conditions in Eq. (2.1.16), we obtain

$$a_{\mu 0} = 0, \quad a_{i3} = 0, \quad a_{11} + a_{22} = 0 \quad (i = 1, 2, 3).$$

By defining $a_{11} = a_+$ and $a_{12} = a_\times$, the solution can be written as

$$h_{\mu\nu}^{\text{TT}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_+ & a_\times & 0 \\ 0 & a_\times & -a_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \exp[-ik(ct - z)]. \quad (2.1.18)$$

This result shows that gravitational waves are transverse waves propagating at the speed of light and possess two independent polarization states, known as the $+$ and $\times$ modes.

Finally, we examine the effect of gravitational waves on freely falling test particles. As before, we consider a monochromatic plane wave propagating in the z -direction. Letting the angular frequency be $\omega = kc$, the spacetime interval is written as

$$\begin{aligned} ds^2 &= (\eta_{\mu\nu} + h_{\mu\nu}^{\text{TT}}) dx^\mu dx^\nu \\ &= -(cdt)^2 + [1 + a_+ \cos(-\omega t + kz)] dx^2 + [1 - a_+ \cos(-\omega t + kz)] dy^2 \\ &\quad + 2a_\times \cos(-\omega t + kz) dx dy + dz^2. \end{aligned} \quad (2.1.19)$$

From this expression, it is clear that the $+$ mode periodically stretches and squeezes distances along the x and y axes, while the $\times$ mode produces a similar deformation along axes rotated by $\pi/4$ with respect to the x and y axes. The resulting motion of freely falling test particles for each polarization mode is illustrated in Fig. 2.1.

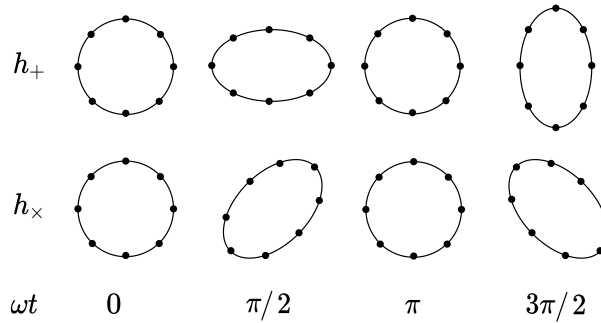


Figure 2.1: Deformation of free particles induced by gravitational waves with different polarization modes.

2.1.3 Astrophysical sources

In this subsection, we introduce astrophysical events and objects that are expected to generate gravitational waves. Figure 2.2 shows the characteristic strain amplitudes and typical frequency bands of various gravitational-wave sources. Currently predicted gravitational-wave sources include the following:

- **Compact binary coalescences** — Mergers of compact binaries, such as binary black holes (BBHs) and binary neutron stars (BNSs), are among the most prominent sources of gravitational waves. These events are collectively referred to as compact binary coalescences (CBCs). All gravitational-wave signals detected to date originate from CBCs, with the majority being BBH mergers. The gravitational-wave signal from a CBC can be divided into three phases: the inspiral, merger, and ringdown

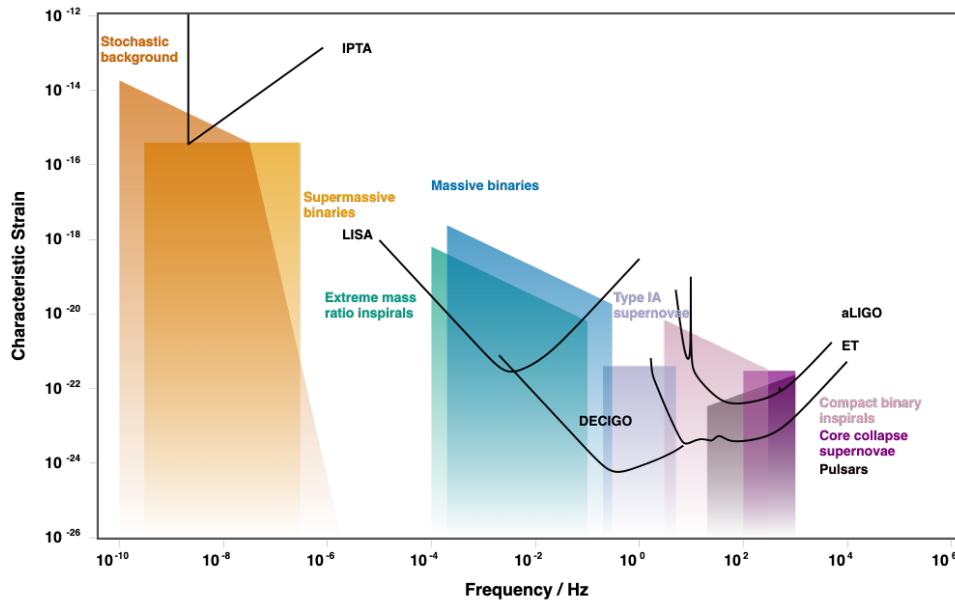


Figure 2.2: Sensitivity curves of gravitational-wave detectors and typical gravitational-wave sources [12].

phases. During the inspiral phase, the binary components gradually approach each other, leading to an increase in the gravitational-wave frequency over time and producing a characteristic chirp signal. The merger phase corresponds to the coalescence of the two compact objects and generates the strongest gravitational-wave emission. In the ringdown phase, the remnant object emits gravitational waves through damped oscillations as it settles into a stable configuration. For BBH mergers, the ringdown signal can be described by quasinormal modes (QNMs). By measuring the frequencies and decay rates of these modes, the mass and spin of the remnant black hole can be inferred. This approach, known as black hole spectroscopy [13], enables tests of general relativity and constraints on alternative theories of gravity.

- **Massive binaries** — Supermassive binary systems are also expected to be sources of gravitational waves. Their typical frequencies are predicted to lie in the millihertz band or below. Consequently, their detection is expected not with current ground-based detectors, but with space-based gravitational-wave observatories such as LISA and DECIGO.
- **Pulsars** — Pulsars are rotating neutron stars that emit electromagnetic radiation. If a pulsar possesses a non-axisymmetric mass distribution, it continuously emits gravitational waves [14].
- **Supernova explosions** — Core-collapse supernovae are known to generate gravitational waves. These explosions occur when massive stars reach the end of their lifetimes and collapse gravitationally, eventually forming either a neutron star or a black hole. Although no gravitational waves from core-collapse supernovae have been

observed so far, they are expected to emit gravitational waves with frequencies of order ~ 1 kHz. Successful detection would provide valuable insights into the explosion mechanism.

- **Stochastic gravitational-wave background** — A random superposition of weak signals originating from a large number of unresolved sources forms the stochastic gravitational-wave background. This signal can be detected through cross-correlation analyses between multiple detectors and is expected to provide a powerful probe of the early Universe [15].

2.2 Interferometric Gravitational-Wave Detectors

Various types of gravitational-wave detectors have been proposed and developed, including laser interferometric detectors, resonant-mass detectors, Doppler tracking experiments, torsion-bar antennas [16], and pulsar timing arrays such as the International Pulsar Timing Array (IPTA), which detect gravitational waves through variations in the arrival times of electromagnetic pulses emitted by pulsars [17]. In this chapter, we focus on the principles of laser interferometric gravitational-wave detectors, represented by ground-based observatories such as LIGO, Virgo, and KAGRA.

2.2.1 Michelson interferometer

The basic configuration of a laser interferometric gravitational-wave detector is the Michelson interferometer. A schematic diagram of a Michelson interferometer is shown in Fig. 2.3. The operating principle of the Michelson interferometer is described as follows.

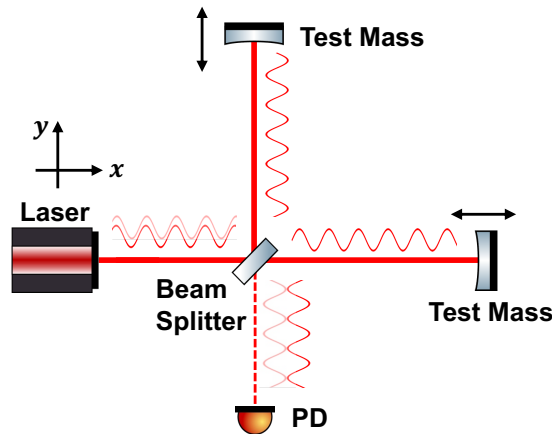


Figure 2.3: Schematic of a Michelson interferometer.

First, laser light emitted from a laser source is split by a beam splitter (BS) into two orthogonal optical paths. Each beam is then reflected by a mirror at the end of the arm and returns to the beam splitter, where the two beams recombine. As a result, interference signals appear both in the direction of the photodetector (PD) and back toward the laser. By monitoring changes in the interference pattern, variations in the arm lengths of the interferometer induced by gravitational waves can be detected. This constitutes the fundamental principle of laser interferometric gravitational-wave detection.

Next, we derive the frequency response of a Michelson interferometer to an incident gravitational wave. We adopt the transverse-traceless (TT) gauge and assume that one arm of the interferometer lies along the x -axis and the other along the y -axis, as shown in Fig. 2.3. The arm lengths are denoted by L_x and L_y , respectively. We consider a $+$ -polarized gravitational wave $h(t)$ with angular frequency Ω propagating in the z -direction. Since the spacetime interval for a photon propagating along the arm is null, we have

$$ds^2 = -c^2 dt^2 + [1 + h(t)] dx^2 = 0. \quad (2.2.1)$$

Assuming $|h(t)| \ll 1$ and $dx/dt > 0$, this leads to

$$\left[1 - \frac{1}{2}h(t)\right] c dt = dx. \quad (2.2.2)$$

Let Δt_x denote the round-trip travel time of light along the x -arm. Integrating the above equation from $t - \Delta t_x$ to t , we obtain

$$\begin{aligned} \Delta t_x &= \frac{2L_x}{c} + \frac{1}{2} \int_{t-\Delta t_x}^t h(t') dt' \\ &\simeq \frac{2L_x}{c} + \frac{1}{2} \int_{t-\frac{2L_x}{c}}^t h(t') dt'. \end{aligned} \quad (2.2.3)$$

If the angular frequency of the laser light is ω_0 , the corresponding phase shift accumulated during the round trip is given by $\phi_x = \omega_0 \Delta t_x$, yielding

$$\phi_x(t) = \frac{2L_x \omega_0}{c} + \frac{\omega_0}{2} \int_{t-\frac{2L_x}{c}}^t h(t') dt'. \quad (2.2.4)$$

Similarly, the phase shift for the y -arm is

$$\phi_y(t) = \frac{2L_y \omega_0}{c} - \frac{\omega_0}{2} \int_{t-\frac{2L_y}{c}}^t h(t') dt'. \quad (2.2.5)$$

Assuming nearly equal arm lengths $L \simeq L_x \simeq L_y$, the phase difference between the two returning beams, $\phi_-(t) = \phi_x(t) - \phi_y(t)$, becomes

$$\begin{aligned} \phi_-(t) &= \frac{2(L_x - L_y)\omega_0}{c} + \omega_0 \int_{t-\frac{2L}{c}}^t h(t') dt' \\ &:= \frac{2(L_x - L_y)\omega_0}{c} + \delta\phi_{\text{GW}}(t). \end{aligned} \quad (2.2.6)$$

The first term represents the static phase difference in the absence of gravitational waves, while the second term corresponds to the phase shift induced by the gravitational wave. Thus, the gravitational wave causes a modulation of the interference signal by an amount $\delta\phi_{\text{GW}}(t)$.

Introducing the Fourier transform of $h(t)$ as

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(\Omega) e^{i\Omega t} d\Omega, \quad (2.2.7)$$

the frequency response of the gravitational-wave-induced phase shift is obtained as

$$\begin{aligned}\delta\phi_{\text{GW}}(t) &= \omega_0 \int_{t-\frac{2L}{c}}^t \int_{-\infty}^{\infty} h(\Omega) e^{i\Omega t'} d\Omega dt' \\ &= \int_{-\infty}^{\infty} \frac{2\omega_0}{\Omega} \sin\left(\frac{\Omega L}{c}\right) e^{-i\frac{\Omega L}{c}} h(\Omega) e^{i\Omega t} d\Omega.\end{aligned}\quad (2.2.8)$$

Comparing this expression with the Fourier representation

$$\delta\phi_{\text{GW}}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta\phi_{\text{GW}}(\Omega) e^{i\Omega t} d\Omega, \quad (2.2.9)$$

we find

$$\begin{aligned}H_{\text{MI}}(\Omega) &:= \frac{2\omega_0}{\Omega} \sin\left(\frac{\Omega L}{c}\right) e^{-i\frac{\Omega L}{c}}, \\ \delta\phi_{\text{GW}}(\Omega) &= H_{\text{MI}}(\Omega) h(\Omega).\end{aligned}\quad (2.2.10)$$

Here, $H_{\text{MI}}(\Omega)$ represents the frequency response function of the Michelson interferometer. Once the laser wavelength and arm length are specified, the response to a gravitational wave with angular frequency Ω is fully determined.

From this expression, the optimal arm length L for detecting gravitational waves with angular frequency Ω is given by

$$L = \frac{\pi c}{2\Omega}. \quad (2.2.11)$$

For example, detecting gravitational waves at a frequency of 100 Hz would require an optimal arm length of approximately 750 km. However, due to the curvature of the Earth, constructing an interferometer of such length would require excavating tunnels up to a depth of about 11 km, which is technically impractical. This limitation motivates the introduction of optical resonators into the arms of the Michelson interferometer.

2.2.2 Fabry–Pérot cavity

A Fabry–Pérot cavity is a simple optical cavity formed by two facing mirrors, in which light undergoes multiple reflections and constructively interferes under a resonance condition. This configuration effectively increases the optical path length inside the cavity and allows the intracavity optical field to build up through repeated round trips. In gravitational-wave detectors such as LIGO and KAGRA, Fabry–Pérot cavities are employed in the arms of the Michelson interferometer to enhance the phase response to gravitational waves. A schematic illustration of a Fabry–Pérot cavity is shown in Fig. 2.4.

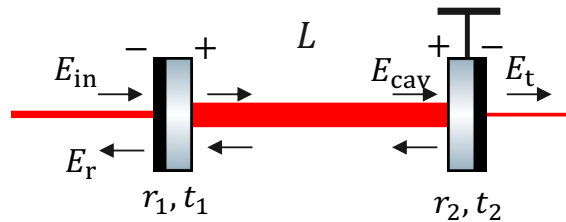


Figure 2.4: Schematic of a Fabry–Pérot cavity.

We denote the amplitude transmissivity and reflectivity of the input test mass (ITM) by t_1 and r_1 , and those of the end test mass (ETM) by t_2 and r_2 .

$$r_1^2 + t_1^2 = 1, \quad r_2^2 + t_2^2 = 1. \quad (2.2.12)$$

Let L be the cavity length. The phase shift accumulated during a single pass through the cavity is then given by $\phi = \omega_0 L/c$.

When an incident field with amplitude E_{in} enters the cavity, the reflected field E_r and transmitted field E_t are given by

$$\begin{aligned} E_r &= -r_1 E_{\text{in}} + t_1^2 r_2 e^{-2i\phi} \sum_{n=0}^{\infty} (r_1 r_2 e^{-2i\phi})^n E_{\text{in}} \\ &= \frac{-r_1 + r_2 e^{-2i\phi}}{1 - r_1 r_2 e^{-2i\phi}} E_{\text{in}}, \\ E_t &= t_1 t_2 e^{-i\phi} \sum_{n=0}^{\infty} (r_1 r_2 e^{-2i\phi})^n E_{\text{in}} \\ &= \frac{t_1 t_2 e^{-i\phi}}{1 - r_1 r_2 e^{-2i\phi}} E_{\text{in}}. \end{aligned} \quad (2.2.13)$$

The intracavity field E_{cav} is similarly expressed as

$$\begin{aligned} E_{\text{cav}} &= t_1 e^{-i\phi} \sum_{n=0}^{\infty} (r_1 r_2 e^{-2i\phi})^n E_{\text{in}} \\ &= \frac{t_1 e^{-i\phi}}{1 - r_1 r_2 e^{-2i\phi}} E_{\text{in}}. \end{aligned} \quad (2.2.14)$$

Since the optical power is proportional to the squared magnitude of the electric field, defining $P_{\text{in}} \propto |E_{\text{in}}|^2$, the reflected, transmitted, and intracavity powers are given by

$$\begin{aligned} P_r &= \frac{r_1^2 + r_2^2 - 2r_1 r_2 \cos(2\phi)}{1 + r_1^2 r_2^2 - 2r_1 r_2 \cos(2\phi)} P_{\text{in}}, \\ P_t &= \frac{t_1^2 t_2^2}{1 + r_1^2 r_2^2 - 2r_1 r_2 \cos(2\phi)} P_{\text{in}}, \\ P_{\text{cav}} &= \frac{t_1^2}{1 + r_1^2 r_2^2 - 2r_1 r_2 \cos(2\phi)} P_{\text{in}}. \end{aligned} \quad (2.2.15)$$

From Eqs. (2.2.2) and (2.2.15), it follows that $P_{\text{in}} = P_r + P_t$ is always satisfied. When the resonance condition $\phi \equiv 0 \pmod{2\pi}$ is met, the intracavity power P_{cav} is maximized. Figure 2.5 shows the dependence of the optical power on the phase shift ϕ for $r_1^2 = 0.95$ and $r_2^2 = 0.95$, with all powers normalized by P_{in} .

Next, we introduce three quantities that characterize the performance of a Fabry–Pérot cavity: the finesse, the free spectral range, and the linewidth. The free spectral range (FSR) is defined as the spacing between adjacent resonance conditions in either cavity length or

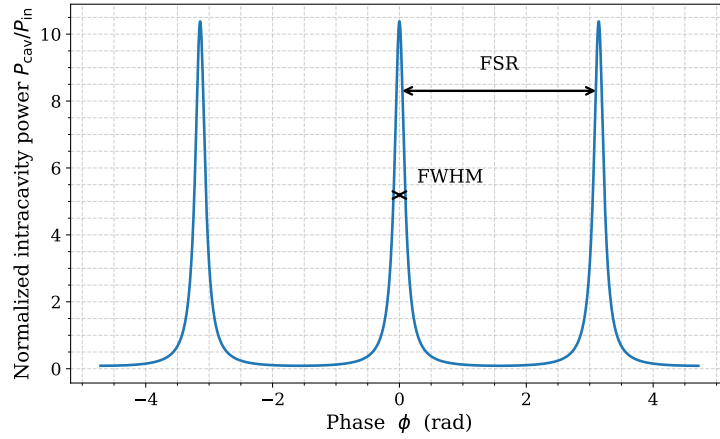


Figure 2.5: Intracavity power of a Fabry–Pérot cavity.

laser frequency:

$$\begin{aligned} L_{\text{FSR}} &:= \frac{\lambda_0}{2}, \\ f_{\text{FSR}} &:= \frac{c}{2L}. \end{aligned} \quad (2.2.16)$$

The linewidth is defined as the full width at half maximum (FWHM) of the resonance peak. In the context of gravitational-wave detectors, the linewidth of the arm cavities is often referred to as the bandwidth, as it determines the observation band. Assuming $f_{\text{FSR}} \gg f_{\text{FWHM}}$, the FWHM is given by

$$f_{\text{FWHM}} = \frac{1 - r_1 r_2}{\pi \sqrt{r_1 r_2}} f_{\text{FSR}}. \quad (2.2.17)$$

The ratio of the free spectral range to the linewidth is called the finesse $\mathcal{F}$, which is expressed as

$$\mathcal{F} = \frac{f_{\text{FSR}}}{f_{\text{FWHM}}} = \frac{\pi \sqrt{r_1 r_2}}{1 - r_1 r_2}. \quad (2.2.18)$$

Near resonance ($\phi \simeq 0$) and for highly reflective mirrors with $r_1, r_2 \simeq 1$, the reflectivities can be approximated as

$$r_i \simeq 1 - \frac{T_i}{2} - \frac{T_i^2}{8} \quad (i = 1, 2). \quad (2.2.19)$$

Under this approximation, the intracavity power given in Eq. (2.2.15) can be approximated by a Lorentzian function:

$$P_{\text{cav}} = \frac{T_1^2}{\frac{1}{4}(T_1 + T_2)^2 + 4\phi^2} P_{\text{in}}. \quad (2.2.20)$$

Here, T_1 and T_2 denote the intensity transmissivities of the mirrors, defined as $T_1 = t_1^2$ and $T_2 = t_2^2$.

2.2.3 Configuration of current detectors

Most current gravitational-wave detectors employ multiple optical cavities in addition to the arm cavities of the interferometer. One representative optical configuration of modern laser interferometric gravitational-wave detectors is the dual-recycled Fabry–Pérot Michelson interferometer (DRFPMI). A schematic overview of this configuration is shown in Fig. 2.6. In this configuration, Fabry–Pérot cavities are implemented in both arms of

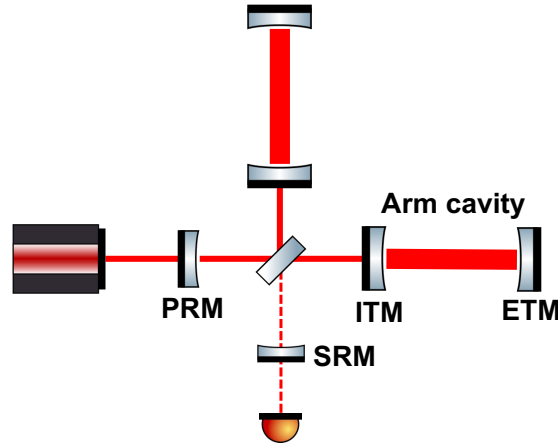


Figure 2.6: Schematic of a dual-recycled Fabry–Pérot Michelson interferometer (DRFPMI).

the Michelson interferometer. In addition, two further cavities are introduced: the power recycling cavity and the signal recycling cavity.

Power recycling

The power recycling cavity (PRC) is formed by placing a power recycling mirror (PRM) at the input port of the interferometer, allowing the light reflected from the interferometer to be reused. This configuration effectively increases the laser power incident on the interferometer and thereby enhances the signal strength.

Signal recycling

The signal recycling cavity (SRC) is formed by placing a signal recycling mirror (SRM) at the output port of the interferometer. It allows the signal and fluctuations appearing at the dark fringe output port to be recycled. As discussed in Sec. 3.4.3, the sensitivity bandwidth of the interferometer can be tuned by adjusting the reflectivity of the signal recycling mirror, enabling the detector to achieve an optimal sensitivity for targeted gravitational-wave sources. By appropriately choosing the resonance condition, the signal recycling cavity reduces the storage time of gravitational-wave signal sidebands inside the interferometer, thereby broadening the sensitivity bandwidth. For this reason, the signal recycling cavity is also referred to as the signal extraction cavity (SEC).

2.3 Noise sources

One important figure of merit for gravitational-wave detectors is the sensitivity to gravitational-wave strain, expressed in terms of the amplitude spectral density of the

signal-to-noise ratio. Figure 2.7 shows the target sensitivity of KAGRA. The sensitivity of gravitational-wave detectors is limited by various sources of noise. In this section, we describe the major noise sources in gravitational-wave detectors.

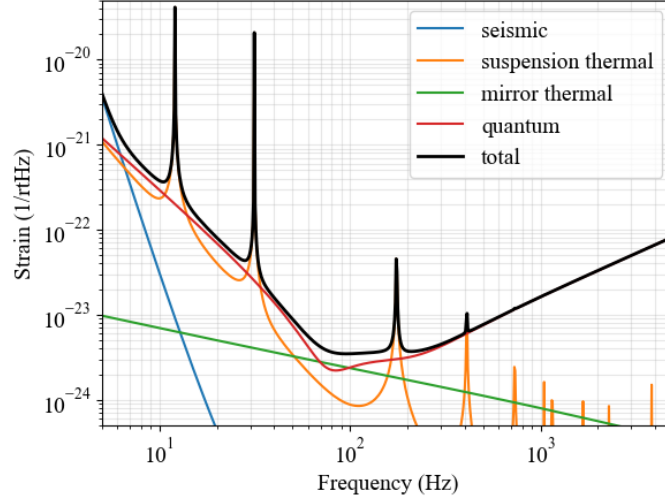


Figure 2.7: Target sensitivity of KAGRA [18].

Power spectral density and amplitude spectral density

We first introduce the power spectral density and the amplitude spectral density, which are commonly used to characterize the sensitivity and noise of gravitational-wave detectors. In noise evaluation, the deviation of a measured quantity from its mean value vanishes when averaged over a sufficiently long time. Therefore, the root mean square (RMS) of the fluctuation is used as a measure of noise.

The frequency-domain representation of the RMS is given by the power spectral density $S(\Omega)$. For a physical quantity $x(t)$, we define its Fourier transform as

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{i\Omega t} d\Omega, \\ X(\Omega) &= \int_{-\infty}^{\infty} x(t) e^{-i\Omega t} dt. \end{aligned} \quad (2.3.1)$$

The power spectral density is then related to the mean square value of $x(t)$ as

$$\langle x^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\Omega) d\Omega. \quad (2.3.2)$$

More explicitly, using the relation $X(-\Omega) = X^*(\Omega)$, we obtain

$$\begin{aligned} \langle x^2 \rangle &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} X(\Omega) X^*(\Omega) d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\Omega) d\Omega, \end{aligned} \quad (2.3.3)$$

which corresponds to Parseval's theorem. Thus, the power spectral density is defined as

$$S(\Omega) := \lim_{T \rightarrow \infty} \frac{|X(\Omega)|^2}{T}. \quad (2.3.4)$$

The square root of the power spectral density is called the amplitude spectral density, denoted by $h_n(\Omega)$:

$$h_n(\Omega) := \sqrt{S(\Omega)}. \quad (2.3.5)$$

The amplitude spectral density represents the frequency-dependent RMS noise. A signal with angular frequency Ω can be detected if its amplitude exceeds $h_n(\Omega)$. From this definition, the unit of the amplitude spectral density is the unit of the physical quantity $x(t)$ multiplied by $1/\sqrt{\text{Hz}}$. Since gravitational-wave strain is defined as the displacement δL normalized by the arm length L , it is dimensionless. Therefore, the amplitude spectral density of gravitational-wave strain has the unit $1/\sqrt{\text{Hz}}$.

2.3.1 Shot noise

Shot noise arises from fluctuations in the number of photons in the laser field. It originates from the statistical nature of photon detection and is intrinsically a frequency-independent white noise. However, in detectors such as KAGRA, which employ optical cavities in the interferometer arms, high-frequency signal components are attenuated by the cavity response, causing shot noise to limit the detector sensitivity at high frequencies.

For a Michelson interferometer operated at a dark fringe, the amplitude spectral density of mirror displacement due to shot noise is given by

$$\delta x_{\text{shot}} = \sqrt{\frac{\hbar c \lambda}{4\pi P}}, \quad (2.3.6)$$

where λ is the laser wavelength and P is the laser power. As evident from this expression, shot noise scales as $1/\sqrt{P}$, and can therefore be reduced by increasing the laser power.

2.3.2 Radiation pressure noise

Radiation pressure noise arises from fluctuations in the photon number, which cause fluctuations in the radiation pressure exerted on the mirrors. This noise limits the detector sensitivity mainly in the frequency band below approximately 100 Hz. For a mirror with mass M , the amplitude spectral density of mirror displacement due to radiation pressure noise is given by

$$\delta x_{\text{rp}} = \frac{1}{M(2\pi f)^2} \sqrt{\frac{16\pi \hbar P}{c\lambda}}, \quad (2.3.7)$$

where P is the laser power and λ is the laser wavelength. As shown by this expression, radiation pressure noise is proportional to $\sqrt{P}$. In contrast to shot noise, increasing the laser power enhances radiation pressure noise. Shot noise and radiation pressure noise, both originating from photon number fluctuations, are collectively referred to as quantum noise.

2.3.3 Thermal noise

Thermal noise arises from energy exchange between the mirrors or suspension system and the surrounding thermal bath, and primarily limits the detector sensitivity in the frequency range from approximately 10 Hz to 100 Hz. The power spectral density of thermal noise can be derived using the fluctuation–dissipation theorem and is expressed in terms of the temperature of the thermal bath and the mechanical impedance of the system. Thermal noise includes several contributions, such as thermoelastic noise caused by non-uniform deformation due to temperature gradients, and Brownian noise associated with the thermal motion of the mirror substrates and optical coatings.

2.3.4 Seismic noise

Seismic noise, caused by ground vibrations, mainly limits the detector sensitivity at low frequencies below approximately 10 Hz. A typical estimate for the amplitude spectral density of mirror displacement due to seismic noise is given by

$$\delta x_{\text{seis}} \sim \left(\frac{1 \text{ Hz}}{f} \right)^2 \times 10^{-7} \text{ m}/\sqrt{\text{Hz}}. \quad (2.3.8)$$

For example, ground vibrations at a frequency of 100 Hz correspond to a displacement amplitude of the order of $10^{-11} \text{ m}/\sqrt{\text{Hz}}$. Therefore, seismic noise must be suppressed using vibration isolation systems. In KAGRA, seismic isolation is achieved using a nine-stage pendulum suspension system.

2.3.5 Newtonian noise

Newtonian noise is caused by fluctuations in the gravitational field generated by moving masses in the environment, such as the atmosphere and the Earth’s crust, which directly perturb the mirrors. Although Newtonian noise is extremely small and has not yet been observed in the spectral density measurements of second-generation detectors, it may limit the low-frequency sensitivity of third-generation gravitational-wave detectors as their sensitivity improves.

Chapter 3

Quantum Noise in Gravitational-Wave Detectors

Quantum noise in gravitational-wave detectors consists of two fundamental contributions: shot noise, which originates from statistical fluctuations in the number of photons measured at the photodetector, and radiation pressure noise, which arises from fluctuations in the radiation pressure exerted on the interferometer mirrors. In this chapter, we introduce the basic concepts of quantum noise in gravitational-wave detectors and formulate its theoretical description.

In Sec. 3.1, we quantize the electromagnetic field and express it in terms of mode functions and creation and annihilation operators, providing a quantum description of electromagnetic fluctuations, including laser light. In Sec. 3.2, typical quantum states of light relevant to laser interferometry, such as photon number states, coherent states, and squeezed states, are introduced. Section 3.3 presents the two-photon formalism, which provides a convenient framework for describing quantum noise in gravitational-wave detectors. As a concrete application, the quantum noise power spectral densities for several optical configurations are calculated in Sec. 3.4. Finally, Sec. 3.5 discusses frequency-dependent squeezing, one of the most important techniques for reducing quantum noise.

3.1 Quantization of the Electromagnetic Field

The propagation of the electromagnetic field in free space, where no charges or currents are present, is governed by Maxwell's equations:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} &= \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}.\end{aligned}\tag{3.1.1}$$

We introduce the vector potential $\mathbf{A}(\mathbf{r}, t)$ as

$$\begin{aligned}\mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t}, \\ \mathbf{B} &= \nabla \times \mathbf{A}.\end{aligned}\tag{3.1.2}$$

Imposing the Coulomb gauge condition, $\nabla \cdot \mathbf{A} = 0$, Maxwell's equations reduce to the wave equation

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = 0. \quad (3.1.3)$$

The vector potential can be expanded in terms of mode functions $\mathbf{u}_k(\mathbf{r})$ determined by the boundary conditions of the system:

$$\mathbf{A}(\mathbf{r}, t) = \sum_k \left(c_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t} + c_k^* \mathbf{u}_k^*(\mathbf{r}) e^{i\omega_k t} \right), \quad (3.1.4)$$

where c_k are complex mode amplitudes. The mode functions satisfy

$$\left(\nabla^2 + \frac{\omega_k^2}{c^2} \right) \mathbf{u}_k(\mathbf{r}) = 0, \quad (3.1.5)$$

together with the transverse condition and the orthonormality relation

$$\begin{aligned} \nabla \cdot \mathbf{u}_k(\mathbf{r}) &= 0, \\ \int_V \mathbf{u}_k^*(\mathbf{r}) \cdot \mathbf{u}_{k'}(\mathbf{r}) d\mathbf{r} &= \delta_{kk'}. \end{aligned} \quad (3.1.6)$$

As a simple example, we consider plane-wave modes in a cubic volume of side length l . The wave vector is given by

$$\mathbf{k} = \frac{2\pi}{l} (n_x, n_y, n_z), \quad (3.1.7)$$

where n_x, n_y, n_z are integers. Due to the Coulomb gauge condition, the polarization vectors $\hat{\mathbf{e}}_\mu$ ($\mu = 1, 2$) satisfy

$$\mathbf{k} \cdot \hat{\mathbf{e}}_\mu = 0, \quad \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = 0. \quad (3.1.8)$$

The corresponding mode functions are

$$\mathbf{u}_k(\mathbf{r}) = l^{-3/2} e^{i\mathbf{k} \cdot \mathbf{r}} \hat{\mathbf{e}}_\mu. \quad (3.1.9)$$

In this case, the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$ are given by

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \frac{1}{l^{3/2}} \sum_{k, \mu} i\omega_k \left(c_{k\mu} e^{-i(\omega_k t - \mathbf{k} \cdot \mathbf{r})} - c_{k\mu}^* e^{i(\omega_k t - \mathbf{k} \cdot \mathbf{r})} \right) \hat{\mathbf{e}}_\mu \\ \mathbf{B}(\mathbf{r}, t) &= \frac{1}{l^{3/2}} \sum_{k, \mu} ik \left(c_{k\mu} e^{-i(\omega_k t - \mathbf{k} \cdot \mathbf{r})} - c_{k\mu}^* e^{i(\omega_k t - \mathbf{k} \cdot \mathbf{r})} \right) (\hat{\mathbf{k}} \times \hat{\mathbf{e}}_\mu) \end{aligned} \quad (3.1.10)$$

Here, $k = |\mathbf{k}|$ denotes the magnitude of the wave vector, and $\hat{\mathbf{k}}$ is the unit vector in the direction of $\mathbf{k}$. We now calculate the energy of the electromagnetic field and show that it can be expressed in terms of creation and annihilation operators. For a quantization volume $V = l^3$, the electric-field contribution to the energy is

$$\begin{aligned} \int \mathbf{E}^2 d\mathbf{r} &= -\frac{1}{V} \int_V \sum_{k, k'} \sum_{\mu, \nu} \omega_k \omega_{k'} \left(c_{k\mu} e^{-i(\omega_k t - \mathbf{k} \cdot \mathbf{r})} - c_{k\mu}^* e^{i(\omega_k t - \mathbf{k} \cdot \mathbf{r})} \right) \\ &\quad \times \left(c_{k'\nu} e^{-i(\omega_{k'} t - \mathbf{k}' \cdot \mathbf{r})} - c_{k'\nu}^* e^{i(\omega_{k'} t - \mathbf{k}' \cdot \mathbf{r})} \right) \hat{\mathbf{e}}_\mu \cdot \hat{\mathbf{e}}_\nu d\mathbf{r}, \end{aligned} \quad (3.1.11)$$

which can be rewritten as

$$\begin{aligned} \int_V \mathbf{E}^2 d\mathbf{r} = & -\frac{1}{V} \int_V \sum_{k,k'} \sum_{\mu,\nu} \omega_k \omega_{k'} \left(c_{k\mu} c_{k'\nu} e^{-i[(\omega_k + \omega_{k'})t - (\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}]} \right. \\ & + c_{k\mu}^* c_{k'\nu}^* e^{i[(\omega_k + \omega_{k'})t - (\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}]} \\ & - c_{k\mu} c_{k'\nu}^* e^{-i[(\omega_k - \omega_{k'})t - (\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}]} \\ & \left. - c_{k\mu}^* c_{k'\nu} e^{i[(\omega_k - \omega_{k'})t - (\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}]} \right) \delta_{\mu\nu} d\mathbf{r}. \end{aligned} \quad (3.1.12)$$

Using the orthogonality relation for plane-wave modes in a cubic volume

$$\int_V e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}} d\mathbf{r} = V \delta_{\mathbf{k}\mathbf{k}'}, \quad (3.1.13)$$

only the terms with $\mathbf{k} = \pm \mathbf{k}'$ survive as a result of the spatial integration. As a result, we obtain

$$\int_V \mathbf{E}^2 d\mathbf{r} = \sum_{k,\mu} \left[-\omega_k \omega_{-k} \left(c_{k\mu} c_{-k\mu} e^{-i(\omega_k + \omega_{-k})t} + c_{k\mu}^* c_{-k\mu}^* e^{i(\omega_k + \omega_{-k})t} \right) + 2\omega_k^2 c_{k\mu} c_{k\mu}^* \right]. \quad (3.1.14)$$

By an analogous calculation, the magnetic-field contribution to the energy is

$$\int_V \mathbf{B}^2 d\mathbf{r} = \sum_{k,\mu} \left[k^2 \left(c_{k\mu} c_{-k\mu} e^{-i(\omega_k + \omega_{-k})t} + c_{k\mu}^* c_{-k\mu}^* e^{i(\omega_k + \omega_{-k})t} \right) + 2k^2 c_{k\mu} c_{k\mu}^* \right]. \quad (3.1.15)$$

Using Eqs. (3.1.14) and (3.1.15), together with the dispersion relation $\omega_k = ck$, the total energy of the electromagnetic field is finally given by

$$\begin{aligned} H &= \int_V \left(\frac{\varepsilon_0}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2 \right) d\mathbf{r} \\ &= 2\varepsilon_0 \omega_k^2 c_{k\mu} c_{k\mu}^*. \end{aligned} \quad (3.1.16)$$

Therefore, using Eqs. (3.1.14) and (3.1.15) together with the dispersion relation $\omega_k = ck$, the energy of the electromagnetic field can be written as

$$\begin{aligned} H &= \int_V \left(\frac{\varepsilon_0}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2 \right) d\mathbf{r} \\ &= \sum_{k,\mu} 2\varepsilon_0 \omega_k^2 c_{k\mu} c_{k\mu}^*. \end{aligned} \quad (3.1.17)$$

By writing the complex amplitude as $c_{k\mu} = x_{k\mu} + iy_{k\mu}$, the Hamiltonian becomes

$$H = \sum_{k,\mu} 2\varepsilon_0 \omega_k^2 (x_{k\mu}^2 + y_{k\mu}^2). \quad (3.1.18)$$

Thus, for each polarization, the energy of the electromagnetic field is described by two real variables corresponding to the real and imaginary parts of the complex amplitude.

Introducing the variables

$$\begin{aligned} Q_{k\mu} &= 2\sqrt{\varepsilon_0} x_{k\mu}, \\ P_{k\mu} &= 2\omega_k \sqrt{\varepsilon_0} y_{k\mu}, \end{aligned} \quad (3.1.19)$$

the electromagnetic-field energy can be written in the same form as the Hamiltonian of a harmonic oscillator:

$$H = \sum_{k,\mu} \frac{1}{2} (\omega_k^2 Q_{k\mu}^2 + P_{k\mu}^2). \quad (3.1.20)$$

Quantization of the electromagnetic field is achieved by promoting $Q_{k\mu}$ and $P_{k\mu}$ to operators satisfying the canonical commutation relation

$$[\hat{Q}_{k\mu}, \hat{P}_{k'\nu}] = i\hbar \delta_{kk'} \delta_{\mu\nu}. \quad (3.1.21)$$

Defining the annihilation and creation operators as

$$\begin{aligned} \hat{a}_{k\mu} &= \sqrt{\frac{\omega_k}{2\hbar}} \hat{Q}_{k\mu} + i \frac{1}{\sqrt{2\hbar\omega_k}} \hat{P}_{k\mu}, \\ \hat{a}_{k\mu}^\dagger &= \sqrt{\frac{\omega_k}{2\hbar}} \hat{Q}_{k\mu} - i \frac{1}{\sqrt{2\hbar\omega_k}} \hat{P}_{k\mu}, \end{aligned} \quad (3.1.22)$$

these operators satisfy the commutation relations

$$\begin{aligned} [\hat{a}_{k\mu}, \hat{a}_{k'\nu}] &= [\hat{a}_{k\mu}^\dagger, \hat{a}_{k'\nu}^\dagger] = 0, \\ [\hat{a}_{k\mu}, \hat{a}_{k'\nu}^\dagger] &= \delta_{kk'} \delta_{\mu\nu}. \end{aligned} \quad (3.1.23)$$

The Hamiltonian of the electromagnetic field is then given by

$$\begin{aligned} \hat{H} &= \sum_{k,\mu} \hbar\omega_k \left(\hat{a}_{k\mu}^\dagger \hat{a}_{k\mu} + \frac{1}{2} \right) \\ &= \sum_{k,\mu} \hbar\omega_k \left(\hat{n}_{k\mu} + \frac{1}{2} \right), \end{aligned} \quad (3.1.24)$$

where $\hat{n}_{k\mu} = \hat{a}_{k\mu}^\dagger \hat{a}_{k\mu}$ is the number operator. From this expression, it is clear that the energy of the electromagnetic field consists of the product of the photon number in each mode and the energy per photon $\hbar\omega_k$, together with an additional contribution $\frac{1}{2}\hbar\omega_k$ arising from vacuum fluctuations.

The quantized electric and magnetic fields are expressed in terms of the creation and annihilation operators as

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= i \sum_{k,\mu} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} \left[a_{k\mu} e^{-i(\omega_k t - \mathbf{k} \cdot \mathbf{r})} - a_{k\mu}^\dagger e^{i(\omega_k t - \mathbf{k} \cdot \mathbf{r})} \right] \hat{\mathbf{e}}_\mu, \\ \mathbf{B}(\mathbf{r}, t) &= i \sum_{k,\mu} \sqrt{\frac{\mu_0 \hbar\omega_k}{2V}} \left[a_{k\mu} e^{-i(\omega_k t - \mathbf{k} \cdot \mathbf{r})} - a_{k\mu}^\dagger e^{i(\omega_k t - \mathbf{k} \cdot \mathbf{r})} \right] (\hat{\mathbf{k}} \times \hat{\mathbf{e}}_\mu). \end{aligned} \quad (3.1.25)$$

3.2 Quantum States of Light

3.2.1 Heisenberg Uncertainty Principle

The Heisenberg uncertainty principle describes a fundamental limitation associated with pairs of non-commuting physical observables. Such observables cannot be simultaneously measured with arbitrary precision. For two non-commuting operators $\hat{A}$ and $\hat{B}$, the following relation holds:

$$[\hat{A}, \hat{B}] \neq 0. \quad (3.2.1)$$

From this commutation relation, an uncertainty relation can be derived as

$$V(\hat{A})V(\hat{B}) \geq \frac{1}{4} \left| \langle [\hat{A}, \hat{B}] \rangle \right|^2, \quad (3.2.2)$$

which is known as the Robertson uncertainty relation. When the equality in Eq. (3.2.2) is satisfied, the corresponding quantum state is referred to as a minimum-uncertainty state.

Here, $V(\hat{A})$ denotes the variance of the operator $\hat{A}$, defined as

$$V(\hat{A}) = \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2. \quad (3.2.3)$$

3.2.2 Photon Number States

Photon number states, also known as Fock states, are quantum states defined as eigenstates of the photon number operator $\hat{N} = \hat{a}^\dagger \hat{a}$ for a single-mode electromagnetic field. A photon number state $|n\rangle$ satisfies

$$\hat{N} |n\rangle = n |n\rangle. \quad (3.2.4)$$

We now evaluate the expectation value and variance of the electric field in a photon number state. In Eq. (3.1.25), the electromagnetic field was expressed as a sum over modes characterized by the wave vector and polarization index $(\mathbf{k}, \mu)$ in a cubic volume V . Here, for simplicity, we consider a single-mode electric field with angular frequency ω , propagating along the z -direction and having a single polarization component. The quantized electric field operator is written as

$$\hat{E}(z, t) = i \sqrt{\frac{\hbar \omega}{2 \epsilon_0 V}} \left[\hat{a} e^{-i(\omega t - kz)} - \hat{a}^\dagger e^{i(\omega t - kz)} \right]. \quad (3.2.5)$$

The annihilation and creation operators satisfy the canonical commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0. \quad (3.2.6)$$

Their action on the photon number states is given by

$$\begin{aligned} \hat{a} |n\rangle &= \sqrt{n} |n-1\rangle, \\ \hat{a}^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle, \\ |n\rangle &= \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle. \end{aligned} \quad (3.2.7)$$

The expectation value of the electric field in a photon number state is

$$\bar{E} = \langle n | \hat{E} | n \rangle = 0, \quad (3.2.8)$$

indicating that photon number states carry no information about the phase or amplitude of the electric field.

The variance of the quantized electric field is evaluated as

$$\begin{aligned} V(\hat{E}(z, t)) &= \langle n | (\hat{E} - \bar{E})^2 | n \rangle \\ &= \langle n | \hat{E}^2 | n \rangle \\ &= \frac{\hbar\omega}{2\varepsilon_0 V} (2n + 1). \end{aligned} \quad (3.2.9)$$

Even in the vacuum state corresponding to $n = 0$, the variance of the electric field takes a finite value. This result demonstrates the existence of quantum fluctuations of the electromagnetic field in the vacuum state, which originate from the zero-point energy.

3.2.3 Quadrature Operators

Hermitian operators corresponding to measurable physical quantities can be defined as linear combinations of the annihilation and creation operators as

$$\hat{X}_1 = \hat{a} + \hat{a}^\dagger, \quad \hat{X}_2 = -i(\hat{a} - \hat{a}^\dagger), \quad (3.2.10)$$

where $\hat{X}_1$ and $\hat{X}_2$ are Hermitian operators. Using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, the commutator of these operators is given by

$$[\hat{X}_1, \hat{X}_2] = 2i. \quad (3.2.11)$$

The corresponding uncertainty relation is therefore

$$V(\hat{X}_1)V(\hat{X}_2) \geq 1. \quad (3.2.12)$$

Using the quadrature operators $\hat{X}_1$ and $\hat{X}_2$, the single-mode electric field defined in Eq. (3.2.5) can be rewritten as

$$\begin{aligned} \hat{E}(z, t) &= i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left[\frac{\hat{X}_1 + i\hat{X}_2}{2} e^{-i(\omega t - kz)} - \frac{\hat{X}_1 - i\hat{X}_2}{2} e^{i(\omega t - kz)} \right] \\ &= \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left[\hat{X}_1 \sin(\omega t - kz) - \hat{X}_2 \cos(\omega t - kz) \right]. \end{aligned} \quad (3.2.13)$$

From this expression, $\hat{X}_1$ and $\hat{X}_2$ correspond to the amplitudes of the sine and cosine components of the electric field, respectively, and represent two orthogonal quadrature components. These operators are collectively referred to as quadrature operators. In particular, $\hat{X}_1$ is called the phase quadrature, while $\hat{X}_2$ is referred to as the amplitude quadrature.

3.2.4 Coherent States

A single-mode laser field with sufficiently stabilized intensity and frequency is well described by a coherent state. Coherent states minimize the uncertainty product of non-commuting observables and are therefore referred to as minimum-uncertainty states [19]. A single-mode coherent state $|\alpha\rangle$ is defined as the state obtained by applying the unitary displacement operator $\hat{D}(\alpha)$ to the vacuum state $|0\rangle$:

$$|\alpha\rangle = \hat{D}(\alpha) |0\rangle, \quad \hat{D}(\alpha) = \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right). \quad (3.2.14)$$

The coherent state $|\alpha\rangle$ is a right eigenstate of the annihilation operator,

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle, \quad (3.2.15)$$

and can be expanded in the photon number basis as

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (3.2.16)$$

The photon number distribution of a coherent state follows a Poisson distribution:

$$P(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}, \quad (3.2.17)$$

where the mean photon number is given by $\langle \hat{N} \rangle = |\alpha|^2$. Writing the eigenvalue as $\alpha = |\alpha|e^{i\theta}$, the expectation values of the quadrature operators are

$$\begin{aligned} \langle \alpha | \hat{X}_1 | \alpha \rangle &= \alpha + \alpha^* = 2|\alpha| \cos \theta, \\ \langle \alpha | \hat{X}_2 | \alpha \rangle &= 2|\alpha| \sin \theta. \end{aligned} \quad (3.2.18)$$

We now verify that coherent states are minimum-uncertainty states. The variances of the quadrature operators in a coherent state are given by

$$\begin{aligned} V_\alpha(\hat{X}_1) &= \langle \alpha | \hat{X}_1^2 | \alpha \rangle - \langle \alpha | \hat{X}_1 | \alpha \rangle^2 = 1, \\ V_\alpha(\hat{X}_2) &= \langle \alpha | \hat{X}_2^2 | \alpha \rangle - \langle \alpha | \hat{X}_2 | \alpha \rangle^2 = 1. \end{aligned} \quad (3.2.19)$$

Therefore, the uncertainties in the two quadrature components are equal, and the uncertainty relation is saturated:

$$V_\alpha(\hat{X}_1) V_\alpha(\hat{X}_2) = 1. \quad (3.2.20)$$

These results show that coherent states represent the quantum states closest to a classical electromagnetic field.

3.2.5 Squeezed States

A squeezed state is a quantum state in which the uncertainty of one quadrature is reduced below that of a coherent state, at the expense of increased uncertainty in the orthogonal quadrature, while still satisfying the Heisenberg uncertainty relation. A squeezed

state $|\beta\rangle$ is obtained by applying a squeeze operator to the vacuum or a coherent state:

$$|\beta\rangle = \hat{S}(r, \phi) |0\rangle, \quad \hat{S}(r, \phi) = \exp \left[\frac{1}{2} r e^{-2i\phi} \hat{a}^2 - \frac{1}{2} r e^{2i\phi} \hat{a}^{\dagger 2} \right], \quad (3.2.21)$$

where r is the squeezing factor and ϕ is the squeezing angle. Squeezing applied to the vacuum state and coherent state is referred to as a squeezed vacuum state and a bright squeezed state, respectively.

The transformation of the quadrature operators under the squeeze operator is given by

$$\hat{S}^\dagger(r, \phi) \hat{X}_1 \hat{S}(r, \phi) = \hat{X}_1 (\cosh r - \sinh r \cos 2\phi) - \hat{X}_2 \sinh r \sin 2\phi, \quad (3.2.22)$$

$$\hat{S}^\dagger(r, \phi) \hat{X}_2 \hat{S}(r, \phi) = \hat{X}_2 (\cosh r + \sinh r \cos 2\phi) - \hat{X}_1 \sinh r \sin 2\phi. \quad (3.2.23)$$

These relations can be derived using the Baker–Campbell–Hausdorff formula. In the special case $\phi = 0$, the transformations reduce to

$$\hat{S}^\dagger(r) \hat{X}_1 \hat{S}(r) = \hat{X}_1 e^{-r}, \quad (3.2.24)$$

$$\hat{S}^\dagger(r) \hat{X}_2 \hat{S}(r) = \hat{X}_2 e^r. \quad (3.2.25)$$

The variances of the quadrature operators in a squeezed state are therefore

$$\begin{aligned} V_\beta(\hat{X}_1) &= e^{-2r}, \\ V_\beta(\hat{X}_2) &= e^{2r}, \end{aligned} \quad (3.2.26)$$

indicating that the fluctuation in one quadrature is reduced, while that in the orthogonal quadrature is enhanced. Nevertheless, the product of the variances remains constant:

$$V_\beta(\hat{X}_1) V_\beta(\hat{X}_2) = 1, \quad (3.2.27)$$

showing that squeezed states are also minimum-uncertainty states. The coherent and squeezed states can be visualized using a phase-space diagram in the quadrature plane, as shown in Fig. 3.1. A coherent state exhibits equal uncertainties in both quadratures (amplitude and phase), whereas a squeezed state shows reduced uncertainty in one quadrature at the expense of increased uncertainty in the orthogonal quadrature.

3.3 Two-Photon Formalism

Quantum noise in interferometric gravitational-wave detectors can be described as quantum fluctuations of the electromagnetic field. In this work, we adopt the two-photon formalism developed by Caves and Schumaker [20, 21] to describe the signal and noise response of gravitational-wave detectors. This formalism is particularly useful for analyzing quantum noise correlations in interferometric measurements. To work in the frequency

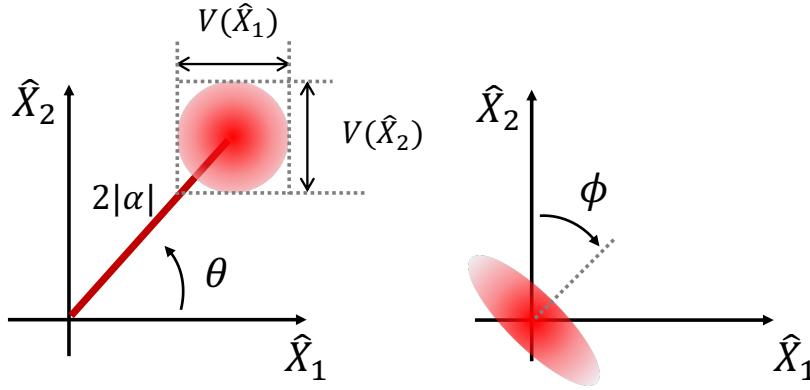


Figure 3.1: Phase-space diagram of a coherent state (left) and a squeezed vacuum state (right). The red line represents the classical field amplitude, while the shaded region indicates quantum fluctuations. θ and ϕ denote the phase of the coherent state and the squeezing angle, respectively.

domain, the Fourier transform of a time-domain operator $\hat{a}(t)$ is defined as

$$\hat{a}(\omega) = \int_{-\infty}^{\infty} \hat{a}(t) e^{i\omega t} dt, \quad (3.3.1)$$

$$\hat{a}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{a}(\omega) e^{-i\omega t} d\omega. \quad (3.3.2)$$

The annihilation and creation operators in the frequency domain satisfy the canonical commutation relations

$$[\hat{a}(\omega), \hat{a}^\dagger(\omega')] = 2\pi\delta(\omega - \omega'), \quad (3.3.3)$$

$$[\hat{a}(\omega), \hat{a}(\omega')] = [\hat{a}^\dagger(\omega), \hat{a}^\dagger(\omega')] = 0. \quad (3.3.4)$$

In the two-photon formalism, the upper and lower sidebands around the carrier frequency ω_0 , namely $\omega_0 \pm \Omega$, are treated as a pair. The corresponding annihilation operators are defined as

$$\hat{a}_+ \equiv \hat{a}(\omega_0 + \Omega), \quad \hat{a}_- \equiv \hat{a}(\omega_0 - \Omega). \quad (3.3.5)$$

These operators satisfy

$$[\hat{a}_\pm(\Omega), \hat{a}_\pm^\dagger(\Omega')] = 2\pi\delta(\Omega - \Omega'), \quad [\hat{a}_+(\Omega), \hat{a}_-^\dagger(\Omega')] = 0. \quad (3.3.6)$$

Using these operators, the two-photon quadrature operators at frequency Ω are defined as

$$\hat{A}_1(\Omega) = \hat{a}_+(\Omega) + \hat{a}_-^\dagger(\Omega), \quad \hat{A}_2(\Omega) = -i(\hat{a}_+(\Omega) - \hat{a}_-^\dagger(\Omega)), \quad (3.3.7)$$

which satisfy the commutation relation

$$[\hat{A}_1(\Omega), \hat{A}_2(\Omega')] = 4\pi i \delta(\Omega - \Omega'). \quad (3.3.8)$$

In the analysis of quantum noise, the optical field is decomposed into a steady classical component and quantum fluctuations, and the system is linearized around the steady state:

$$\hat{\mathbf{A}}(\Omega) = \bar{\mathbf{A}}(\Omega) + \hat{\mathbf{a}}(\Omega), \quad (3.3.9)$$

where $\hat{\mathbf{A}}(\Omega) = (\hat{A}_1(\Omega), \hat{A}_2(\Omega))^T$. We assume $\langle \hat{\mathbf{a}}(\Omega) \rangle = 0$, and that the steady component is much larger than the fluctuations, $|\bar{\mathbf{A}}| \gg |\hat{\mathbf{a}}|$. With this representation, the input–output relations of optical systems can be expressed as linear transformations acting on quadrature vectors at each frequency Ω , allowing a unified treatment of optical elements and quantum noise. In the following, we describe how propagation, reflection/transmission, squeezing, and suspended mirrors act on quadrature vectors as shown in Fig. 3.2.

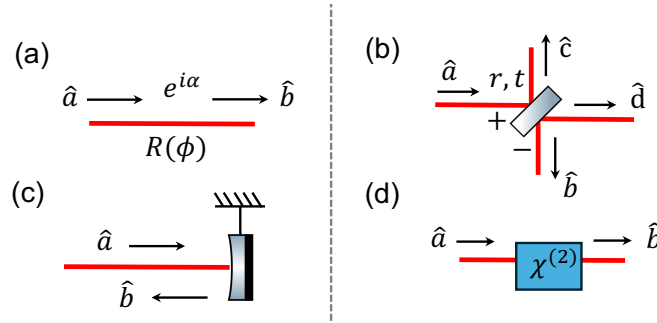


Figure 3.2: Schematic of input–output relations of optical systems.

3.3.1 Propagation

We consider an optical field $\hat{A}(t)$ propagating a distance L in free space. With the propagation time $\tau = L/c$, the output field is given by a time delay,

$$\hat{B}(t) = \hat{A}(t - \tau). \quad (3.3.10)$$

In the two-photon formalism, the optical field is represented by quadrature vectors,

$$\bar{\mathbf{A}}(t) = [\bar{A}_1(t), \bar{A}_2(t)]^T, \quad \hat{\mathbf{a}}(t) = [\hat{a}_1(t), \hat{a}_2(t)]^T. \quad (3.3.11)$$

The propagation over a distance L introduces a carrier phase shift $\phi = \omega_0 L/c$ and is represented as a rotation in quadrature space:

$$\begin{aligned} \bar{\mathbf{B}}(t) &= R(\phi) \bar{\mathbf{A}}(t), \\ \hat{\mathbf{b}}(t) &= R(\phi) \hat{\mathbf{a}}(t - \tau), \end{aligned} \quad (3.3.12)$$

where the rotation matrix is defined as

$$R(\phi) = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}. \quad (3.3.13)$$

In the frequency domain, the sideband phase delay $\alpha = -\Omega L/c$ leads to

$$\begin{aligned}\bar{\mathbf{B}}(\Omega) &= R(\phi)\bar{\mathbf{A}}(\Omega), \\ \hat{\mathbf{b}}(\Omega) &= e^{i\alpha}R(\phi)\hat{\mathbf{a}}(\Omega).\end{aligned}\tag{3.3.14}$$

Thus, propagation is described as a carrier-induced rotation and a frequency-dependent phase delay of the sidebands as shown in Fig. 3.2 (a).

3.3.2 Reflection and Transmission

As shown in Fig. 3.2 (b), consider a symmetric mirror with intensity reflectivity r^2 and transmissivity t^2 , satisfying $r^2 + t^2 = 1$, and assume no optical loss. The transformation of quadrature vectors due to reflection and transmission is given by

$$\begin{aligned}\bar{\mathbf{C}}(\Omega) &= r\bar{\mathbf{A}}(\Omega) + t\bar{\mathbf{B}}(\Omega), \\ \bar{\mathbf{D}}(\Omega) &= t\bar{\mathbf{A}}(\Omega) - r\bar{\mathbf{B}}(\Omega), \\ \hat{\mathbf{c}}(\Omega) &= r\hat{\mathbf{a}}(\Omega) + t\hat{\mathbf{b}}(\Omega), \\ \hat{\mathbf{d}}(\Omega) &= t\hat{\mathbf{a}}(\Omega) - r\hat{\mathbf{b}}(\Omega).\end{aligned}\tag{3.3.15}$$

The relative sign difference accounts for the π phase shift between reflections from the front and back surfaces of the mirror, ensuring unitarity of the beam splitter transformation.

3.3.3 Squeezing

As shown in Fig. 3.2(d), when an optical field passes through a nonlinear crystal, the transformation of the quadrature vectors is described by a squeezing matrix $S(r, \phi)$:

$$\begin{aligned}\mathbf{B} &= S(r, \phi)\mathbf{A}, \\ \mathbf{b} &= S(r, \phi)\mathbf{a}.\end{aligned}\tag{3.3.16}$$

The squeezing matrix is given by

$$S(r, \phi) = \begin{pmatrix} \cosh r - \sinh r \cos 2\phi & \sinh r \sin 2\phi \\ \sinh r \sin 2\phi & \cosh r + \sinh r \cos 2\phi \end{pmatrix}.\tag{3.3.17}$$

For $\phi = 0$, the squeezing matrix reduces to

$$S(r, 0) = \begin{pmatrix} e^{-r} & 0 \\ 0 & e^r \end{pmatrix} \equiv \begin{pmatrix} 1/s & 0 \\ 0 & s \end{pmatrix},\tag{3.3.18}$$

where $s = e^r$ is the squeezing factor and ϕ is the squeezing angle. The squeezing level is often expressed in decibels as

$$\sigma_{\text{dB}} = 20 \log_{10} s.\tag{3.3.19}$$

3.3.4 Ponderomotive Squeezing

In order for a Michelson interferometer to be sensitive to gravitational waves, its mirrors must behave as free masses, and are therefore suspended. As shown in Fig. 3.2 (c), the

radiation-pressure fluctuations exerted on a suspended mirror induce mirror displacement, which in turn modulates the phase quadrature of the reflected field. This process generates correlations between quadratures and is known as ponderomotive squeezing.

When the mirror undergoes a small displacement $\delta x(t)$, the phase of the optical field changes by $2k_0\delta x(t)$. The reflected field $\hat{B}(t)$ is then approximated as

$$\hat{B}(t) \simeq (\bar{A}_1 + \hat{a}_1(t) - 2\bar{A}_2 k_0 \delta x(t)) \cos \omega_0 t + (\bar{A}_2 + \hat{a}_2(t) + 2\bar{A}_1 k_0 \delta x(t)) \sin \omega_0 t. \quad (3.3.20)$$

The corresponding quadrature vectors are

$$\begin{aligned} \bar{\mathbf{B}} &= \bar{\mathbf{A}}, \\ \hat{\mathbf{b}}(t) &= \hat{\mathbf{a}}(t) + 2k_0 \begin{bmatrix} -\bar{A}_2 \\ \bar{A}_1 \end{bmatrix} \delta x(t). \end{aligned} \quad (3.3.21)$$

After Fourier transformation,

$$\hat{\mathbf{b}}(\Omega) = \hat{\mathbf{a}}(\Omega) + 2k_0 \begin{bmatrix} -\bar{A}_2 \\ \bar{A}_1 \end{bmatrix} \delta x(\Omega). \quad (3.3.22)$$

The mirror displacement $\delta x(\Omega)$ is obtained by solving the equation of motion of the mirror,

$$m\Omega^2 \delta x(\Omega) = \delta F_{\text{rp}}(\Omega) + \frac{1}{2}mL\Omega^2 h(\Omega). \quad (3.3.23)$$

where the average radiation pressure force is assumed to be balanced by the mechanical restoring force of the suspended mirror and is therefore neglected. Solving the equation of motion for the mirror displacement, we obtain

$$\delta x(\Omega) = \frac{1}{m\Omega^2} \delta F_{\text{rp}}(\Omega) + \frac{L}{2} h(\Omega). \quad (3.3.24)$$

Radiation pressure fluctuations δF_{rp} arise from fluctuations in the photon number and the momentum transfer per photon, and are given by

$$\delta F_{\text{rp}}(\Omega) = 2\hbar k_0 (\bar{A}_1 \hat{a}_1(\Omega) + \bar{A}_2 \hat{a}_2(\Omega)). \quad (3.3.25)$$

Substituting $\delta x(\Omega)$ into Eq. (3.3.22), the input-output relation becomes

$$\hat{\mathbf{b}}(\Omega) = P(\kappa, \xi_A) \hat{\mathbf{a}}(\Omega) + \frac{\sqrt{2\kappa}}{\hbar_{\text{SQL}}} \mathbf{n}_{A\perp} h(\Omega), \quad (3.3.26)$$

where

$$\mathbf{n}_{A\perp} = \begin{bmatrix} -\sin \xi_A \\ \cos \xi_A \end{bmatrix} \quad (3.3.27)$$

is a unit vector orthogonal to the carrier field direction, and $\xi_A = \arctan(A_2/A_1)$ is the

angle of the carrier field in quadrature space. The matrix $P(\kappa, \xi_A)$ describes the ponderomotive squeezing induced by the radiation-pressure interaction and is given by

$$P(\kappa, \xi_A) \equiv R(\xi_A) \begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix} R(\xi_A), \quad (3.3.28)$$

$$\kappa = \frac{8P_A\omega_0}{mc^2\Omega^2}.$$

Here, κ characterizes the coupling between amplitude and phase fluctuations. h_{SQL} denotes the standard quantum limit for gravitational-wave strain measurements.

$$h_{\text{SQL}} = \sqrt{\frac{8\hbar}{mL^2\Omega^2}}. \quad (3.3.29)$$

In summary, amplitude fluctuations of the optical field are converted into phase fluctuations through radiation-pressure interaction characterized by κ . The gravitational-wave signal appears in the quadrature direction orthogonal to the carrier field. The transformation of the ponderomotive squeezing in Eq. (3.3.28) can be decomposed into rotations and a squeezing matrix as

$$P(\kappa, \xi_A) = R(\xi_A) S(-r, \phi) R(\theta) R(\xi_A), \quad (3.3.30)$$

which makes the squeezing nature of the optomechanical interaction explicit. The squeezing factor r , the squeezing angle ϕ , and the phase rotation angle θ are given by

$$r = \text{arcsinh}(\kappa/2), \quad (3.3.31)$$

$$\phi = \frac{1}{2} \text{arccot}(\kappa/2), \quad (3.3.32)$$

$$\theta = \arctan(\kappa/2). \quad (3.3.33)$$

Since $\kappa \propto 1/\Omega^2$, ponderomotive squeezing exhibits intrinsic frequency dependence. This frequency dependence plays a crucial role in the optimization of quantum noise reduction using frequency-dependent squeezing techniques.

3.4 Quantum Noise in Gravitational-Wave Detectors

In this section, we evaluate the quantum noise of practical laser-interferometric gravitational-wave detectors using the two-photon formalism. By deriving the quantum-noise-limited power spectral density of the signal-to-noise ratio, we obtain the strain sensitivity of the interferometer to gravitational waves.

3.4.1 Quantum Noise of a Michelson Interferometer

Using the quadrature transformations of optical elements derived in the previous section, we calculate the input–output relations of an interferometer and evaluate its quantum noise. We begin with the Michelson interferometer, which represents the most fundamental configuration of gravitational-wave detectors. Figure 3.3 shows a schematic diagram of

the input–output relations of a Michelson interferometer. The beam splitter is assumed to

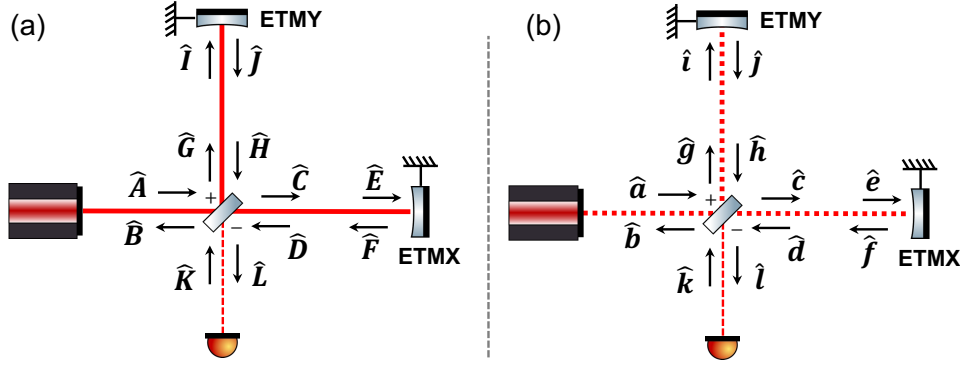


Figure 3.3: Input–output relations of a Michelson interferometer: (a) classical optical fields and (b) quantum fluctuations.

have equal power reflectivity and transmissivity of $1/2$. The arm lengths are denoted by L_x and L_y , and the corresponding carrier phase shifts are $\phi_x = \omega_0 L_x/c$ and $\phi_y = \omega_0 L_y/c$. Assuming nearly equal arm lengths, $L_x \simeq L_y \simeq L$, the sideband phase delay is given by $\alpha = \Omega L/c$. With these assumptions, the input–output relations for the classical optical fields are written as

$$\begin{aligned} B &= \frac{H + D}{\sqrt{2}}, & C &= \frac{A - K}{\sqrt{2}}, & G &= \frac{A + K}{\sqrt{2}}, & L &= \frac{H - D}{\sqrt{2}}, \\ D &= R(\phi_x)F, & E &= R(\phi_x)C, & F &= E, \\ H &= R(\phi_y)J, & I &= R(\phi_y)G, & J &= I. \end{aligned} \quad (3.4.1)$$

We consider the case where the carrier field occupies only the amplitude quadrature, $\mathbf{A} = [A_0, 0]^T$, and no additional fields are incident from the dark port, $\mathbf{K} = \mathbf{0}$. Solving these equations for the output fields $\mathbf{B}$ and $\mathbf{L}$, we obtain

$$\begin{aligned} \mathbf{B} &= \frac{1}{2}R(2\phi_y)[I + R(2\phi_-)]\mathbf{A}, \\ \mathbf{L} &= \frac{1}{2}R(2\phi_x)[I - R(2\phi_-)]\mathbf{A}, \end{aligned} \quad (3.4.2)$$

where $\phi_- = \phi_x - \phi_y$. When $\phi_- = 0$, the output at the antisymmetric (AS) port vanishes, corresponding to the dark fringe condition. In contrast, for $\phi_- = \pi/2$, all optical power exits from the AS port, corresponding to the bright fringe condition.

Next, we consider the input–output relations for quantum fluctuations:

$$\begin{aligned} \hat{b} &= \frac{\hat{h} + \hat{d}}{\sqrt{2}}, & \hat{c} &= \frac{\hat{a} - \hat{k}}{\sqrt{2}}, & \hat{g} &= \frac{\hat{a} + \hat{k}}{\sqrt{2}}, & \hat{l} &= \frac{\hat{h} - \hat{d}}{\sqrt{2}}, \\ \hat{d} &= e^{i\alpha}R(\phi_x)\hat{f}, & \hat{e} &= e^{i\alpha}R(\phi_x)\hat{c}, & \hat{f} &= P(\kappa, \xi_E)\hat{e} - \frac{\sqrt{2\kappa}}{h_{\text{SQL}}^{\text{MI}}}\mathbf{n}_{E\perp}h(\Omega), \\ \hat{h} &= e^{i\alpha}R(\phi_y)\hat{j}, & \hat{i} &= e^{i\alpha}R(\phi_y)\hat{g}, & \hat{j} &= P(\kappa, \xi_J)\hat{i} + \frac{\sqrt{2\kappa}}{h_{\text{SQL}}^{\text{MI}}}\mathbf{n}_{J\perp}h(\Omega). \end{aligned} \quad (3.4.3)$$

For an input power $P_A \propto |A_0|^2$, the circulating power in each arm is $P_{\text{cir}} = P_A/2$. The

coupling constant is defined as $\kappa = 8P_A\omega_0/(mc^2\Omega^2)$, and the standard quantum limit for the Michelson interferometer is $h_{\text{SQL}}^{\text{MI}} = \sqrt{4\hbar/(mL^2\Omega^2)}$. The sign difference of the gravitational-wave terms in $\mathbf{f}$ and $\mathbf{j}$ arises from the quadrupolar nature of gravitational waves, which stretch one arm while compressing the other. For $\mathbf{A} = [A_0, 0]^T$ and $\mathbf{K} = \mathbf{0}$, the angles of carrier field become $\xi_J = \phi_y$ and $\xi_E = \phi_x$, leading to

$$\begin{aligned} P(\kappa, \xi_J) &= R(\phi_y)P(\kappa, 0)R(-\phi_y), \\ P(\kappa, \xi_E) &= R(\phi_x)P(\kappa, 0)R(-\phi_x). \end{aligned} \quad (3.4.4)$$

Using these relations, the output fields $\hat{\mathbf{b}}$ and $\hat{\mathbf{l}}$ are obtained as

$$\begin{aligned} R(-2\phi_x)\hat{\mathbf{b}} &= \frac{e^{2i\alpha}}{2}[I + R(2\phi_-)]P(\kappa, 0)\hat{\mathbf{a}} + \frac{e^{2i\alpha}}{2}[-I + R(2\phi_-)]P(\kappa, 0)\hat{\mathbf{k}} \\ &\quad + \frac{e^{i\alpha}}{2} \frac{\sqrt{2\kappa}}{h_{\text{SQL}}^{\text{MI}}} \begin{bmatrix} -\sin 2\phi_- \\ -1 + \cos 2\phi_- \end{bmatrix} h(\Omega), \\ R(-2\phi_x)\hat{\mathbf{l}} &= \frac{e^{2i\alpha}}{2}[-I + R(2\phi_-)]P(\kappa, 0)\hat{\mathbf{a}} + \frac{e^{2i\alpha}}{2}[I + R(2\phi_-)]P(\kappa, 0)\hat{\mathbf{k}} \\ &\quad + \frac{e^{i\alpha}}{2} \frac{\sqrt{2\kappa}}{h_{\text{SQL}}^{\text{MI}}} \begin{bmatrix} -\sin 2\phi_- \\ 1 + \cos 2\phi_- \end{bmatrix} h(\Omega). \end{aligned} \quad (3.4.5)$$

Solving for $\hat{\mathbf{b}}$ and $\hat{\mathbf{l}}$ under the dark fringe condition $\phi_- = 0$, the gravitational-wave signal appears only in the AS port:

$$\hat{\mathbf{l}}' = e^{2i\alpha} \begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix} \hat{\mathbf{k}} + e^{i\alpha} \frac{\sqrt{2\kappa}}{h_{\text{SQL}}^{\text{MI}}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} h(\Omega), \quad (3.4.6)$$

where $\hat{\mathbf{l}}' = R(-2\phi_x)\hat{\mathbf{l}}$. In this configuration, only vacuum fluctuations entering from the AS port contribute to the quantum noise. The gravitational-wave signal appears in the phase quadrature orthogonal to the carrier field. Defining the noise operator normalized by the signal response as

$$\hat{h}_n = \frac{h_{\text{SQL}}^{\text{MI}}}{\sqrt{2\kappa}}(-\kappa\hat{k}_1 + \hat{k}_2)e^{i\alpha}, \quad (3.4.7)$$

the quantum-noise-limited power spectral density $S_{\text{MI}}(\Omega) = V(\hat{h}_n)$ is given by

$$\begin{aligned} S_{\text{MI}}(\Omega) &= \frac{4\hbar}{mL^2\Omega^2} \left(\frac{8P\omega_0}{mc^2\Omega^2} + \frac{mc^2\Omega^2}{8P\omega_0} \right) \\ &= \frac{(h_{\text{SQL}}^{\text{MI}})^2}{2} \left(\kappa + \frac{1}{\kappa} \right). \end{aligned} \quad (3.4.8)$$

Here, the vacuum fluctuations satisfy $V(\hat{k}_1) = V(\hat{k}_2) = 1$. The first and second terms correspond to the amplitude and phase quadratures, respectively. Due to the frequency dependence of κ , the radiation-pressure noise dominates at low frequencies, while shot noise dominates at high frequencies. Figure 3.4 shows the sensitivity curve of a Michelson interferometer calculated using the parameters listed in Table 3.1. Increasing the laser power reduces shot noise and improves high-frequency sensitivity, while simultaneously

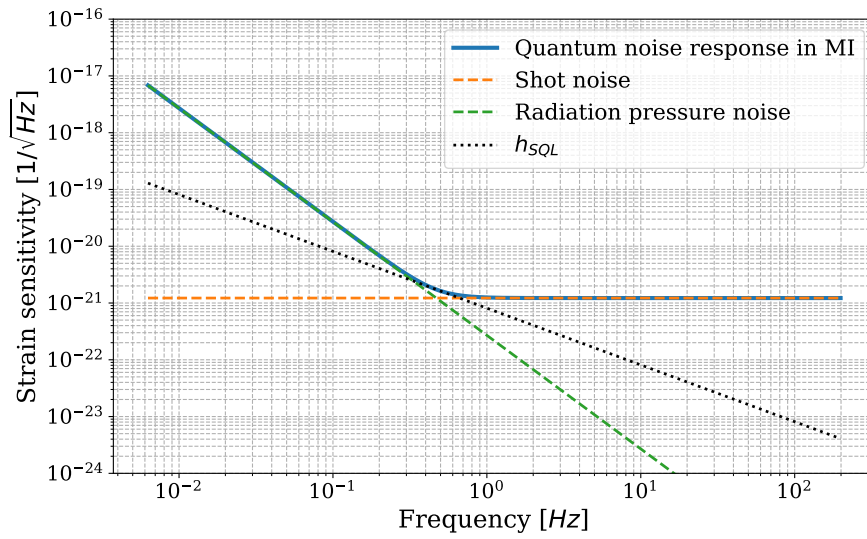


Figure 3.4: Strain sensitivity of a Michelson interferometer.

enhancing radiation-pressure noise and degrading low-frequency sensitivity. As a result, a fundamental trade-off arises between low- and high-frequency quantum noise. This trade-off becomes particularly important for next-generation detectors, in which classical low-frequency noises are significantly suppressed and quantum noise plays a dominant role in determining the overall sensitivity.

3.4.2 Fabry–Pérot Michelson Interferometer and Power Recycling

Most modern ground-based gravitational-wave detectors adopt the dual-recycled Fabry–Pérot Michelson interferometer (DRFPMI), which combines a Fabry–Pérot Michelson interferometer with power recycling and signal recycling. A schematic overview of the DRFPMI configuration is shown in Fig. 3.5.

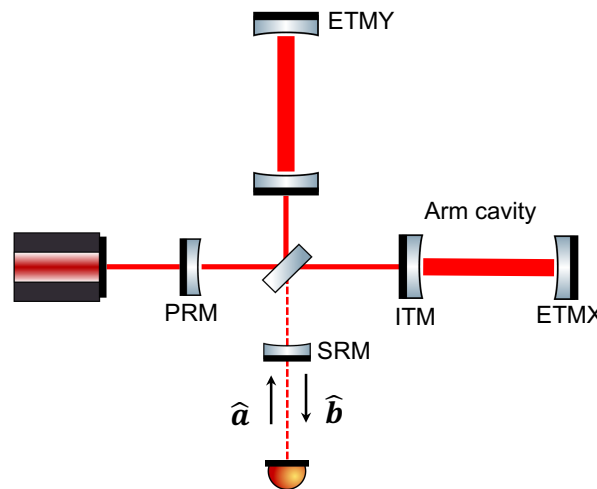


Figure 3.5: Schematic overview of the DRFPMI configuration.

In a Fabry–Pérot Michelson interferometer (FPMI), Fabry–Pérot arm cavities enhance the interferometer response to gravitational waves by effectively increasing the arm length and the circulating optical power. Within the two-photon formalism, the presence of the arm cavities modifies the frequency response of the interferometer, which can be incorporated through the following parameter substitutions:

$$\alpha \rightarrow \beta = \arctan(\Omega/\gamma), \quad (3.4.9)$$

$$\kappa \rightarrow \mathcal{K} = \frac{8P\omega_0}{mLc} \frac{\gamma}{\Omega^2(\gamma^2 + \Omega^2)}, \quad (3.4.10)$$

$$h_{\text{SQL}}^{\text{MI}} \rightarrow h_{\text{SQL}} = \sqrt{\frac{8\hbar}{m\Omega^2 L^2}}. \quad (3.4.11)$$

Here, T_i denotes the power transmissivity of the input test mass of the arm cavity, and the cavity half-linewidth is defined as $\gamma = T_i c / (4L)$. While increasing the cavity finesse strengthens these effects, gravitational-wave sidebands above the cavity linewidth are suppressed by phase delays in the arm cavities, limiting the detection bandwidth. This is known as the storage-time limit. To increase the circulating optical power without further increasing the arm cavity finesse, a power recycling mirror is introduced. When the interferometer is operated at the dark fringe, the Michelson interferometer behaves as a compound mirror with high effective reflectivity. Together with the power-recycling mirror, this forms an optical cavity, allowing the optical power to build up inside the power-recycling cavity. As a result, the effective input power P_{in} incident on the beam splitter is enhanced. Under critical coupling, where the effective reflectivity of this compound mirror matches the reflectivity of the power recycling mirror, the power recycling gain is maximized and is given by

$$G_p = \frac{1}{1 - R_p}, \quad (3.4.12)$$

where R_p is the power reflectivity of the power recycling mirror.

Figure 3.6 shows the sensitivity curves obtained when arm cavities and a power recycling cavity are introduced, using the parameters listed in Table 3.1. Compared to a simple Michelson interferometer, the effective increase in optical power due to the cavities significantly improves the sensitivity, particularly in the high-frequency region.

At low frequencies, however, the sensitivity is degraded by the increase in radiation-pressure noise acting on the test masses.¹ In practice, increasing the optical power is limited by thermal lensing effects in the beamsplitter and mirrors, together with control instabilities.

Furthermore, above the frequency where the SQL is reached, the phase delay induced by the arm cavities leads to destructive interference between the gravitational-wave sidebands, resulting in reduced sensitivity. Thus, while arm cavities improve the best achievable sensitivity, their linewidth γ ultimately limits the detection bandwidth.

¹In second-generation detectors, classical low-frequency noises such as thermal noise and control noise were often dominant over radiation-pressure noise. Therefore, increasing laser power to reduce shot noise at high frequencies was a more important practical consideration.

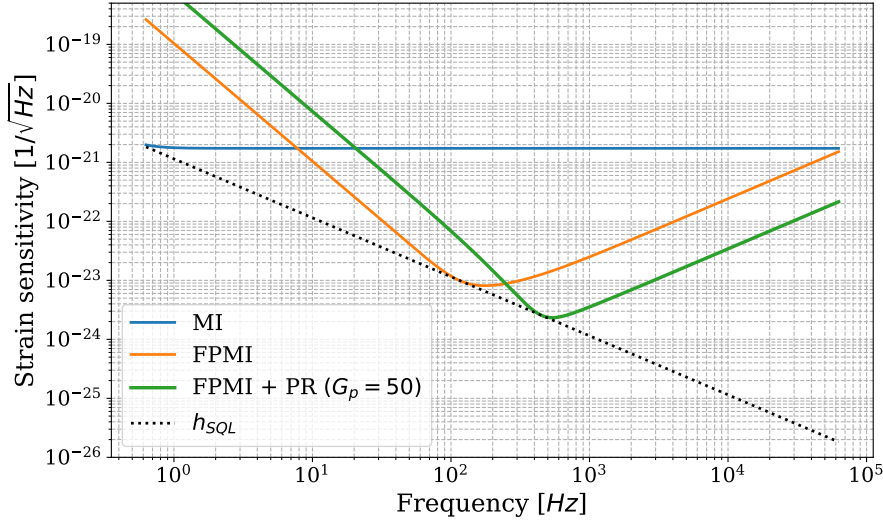


Figure 3.6: Strain sensitivity of a Fabry-Pérot Michelson interferometer (FPMI) with and without power recycling.

3.4.3 Resonant Sideband Extraction

Resonant sideband extraction (RSE) is a well-known technique to overcome this bandwidth limitation imposed by the arm cavities. In RSE, a signal recycling mirror (SRM) placed at the antisymmetric port forms a signal extraction cavity (SEC), which modifies the frequency response of the interferometer. This optical configuration was originally proposed by Mizuno *et al.* [22].

In the following, we denote the SEC length by l_s , and its amplitude reflectivity and transmissivity by ρ and τ , respectively. The input-output relation of the interferometer can be written as

$$\hat{\mathbf{b}} = \frac{1}{M} \left(e^{2i(\beta+\beta_S)} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \hat{\mathbf{a}} + \tau e^{i(\beta+\beta_S)} \frac{\sqrt{2\kappa}}{h_{\text{SQL}}} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} h(\Omega) \right), \quad (3.4.13)$$

where the coefficients are defined as

$$\begin{aligned} M &= 1 + \rho^2 e^{4i(\beta+\beta_S)} - 2\rho e^{2i(\beta+\beta_S)} \left(\cos 2\phi + \frac{\kappa}{2} \sin 2\phi \right), \\ C_{11} &= C_{22} = (1 + \rho^2) \left(\cos 2\phi + \frac{\kappa}{2} \sin 2\phi \right) - 2\rho \cos(2(\beta + \beta_S)), \\ C_{12} &= -\tau^2 (\sin 2\phi + \kappa \sin^2 \phi), \\ C_{21} &= \tau^2 (\sin 2\phi - \kappa \cos^2 \phi), \\ D_1 &= -\left(1 + \rho e^{2i(\beta+\beta_S)} \right) \sin \phi, \\ D_2 &= \left(1 - \rho e^{2i(\beta+\beta_S)} \right) \cos \phi. \end{aligned} \quad (3.4.14)$$

Here, $\phi = \omega_0 l_s / c$ is the carrier phase shift in the SRC, and $\beta_S = \Omega l_s / c$ is the sideband phase delay. We consider a tuned RSE configuration, in which the signal extraction cavity

satisfies the resonance condition $\phi = \pi/2$. In addition, we assume that the signal extraction cavity is sufficiently short so that its phase delay is negligible compared to that of the arm cavities. Under these assumptions, Eq. (3.4.14) reduces to

$$\hat{\mathbf{b}} = \mathbf{T} \hat{\mathbf{a}} + \mathbf{t} h(\Omega), \quad (3.4.15)$$

where the first and second terms represent the quantum noise and the signal response, respectively. Explicitly, they are given by

$$\begin{aligned} \mathbf{T} &= -\frac{1 + \rho^2 + 2\rho \cos 2\beta}{1 + \rho^2 e^{4i\beta} + 2\rho e^{2i\beta}} e^{2i\beta} \begin{bmatrix} 1 & -\frac{\mathcal{K}\tau^2}{1 + \rho^2 + 2\rho \cos 2\beta} \\ 0 & 1 \end{bmatrix}, \\ \mathbf{t} &= -\frac{1 + \rho^2 + 2\rho \cos 2\beta}{1 + \rho^2 e^{4i\beta} + 2\rho e^{2i\beta}} \frac{\sqrt{2\mathcal{K}}}{h_{\text{SQL}}} e^{i\beta} \begin{bmatrix} -1 - \rho e^{2i\beta} \\ 1 + \rho^2 + 2\rho \cos 2\beta \\ 0 \end{bmatrix}. \end{aligned} \quad (3.4.16)$$

The detected output observable is defined as

$$\hat{b}_\zeta = \begin{bmatrix} \cos \zeta & \sin \zeta \end{bmatrix} \hat{\mathbf{b}}, \quad (3.4.17)$$

where ζ is the homodyne angle. The equivalent strain noise $\hat{h}_n(\Omega)$ is defined by normalizing the noise contribution to the signal response,

$$\hat{h}_n \equiv \frac{\begin{bmatrix} \cos \zeta & \sin \zeta \end{bmatrix} \mathbf{T} \hat{\mathbf{a}}}{\begin{bmatrix} \cos \zeta & \sin \zeta \end{bmatrix} \mathbf{t}}. \quad (3.4.18)$$

Assuming vacuum input fields, the single-sided strain noise spectral density is given by

$$S_h = \frac{\begin{bmatrix} \cos \zeta & \sin \zeta \end{bmatrix} \mathbf{T} \mathbf{T}^\dagger \begin{bmatrix} \cos \zeta & \sin \zeta \end{bmatrix}^T}{\begin{bmatrix} \cos \zeta & \sin \zeta \end{bmatrix} \mathbf{t} \mathbf{t}^\dagger \begin{bmatrix} \cos \zeta & \sin \zeta \end{bmatrix}^T}. \quad (3.4.19)$$

For the phase quadrature readout $\zeta = \pi/2$, the quantum noise limited strain sensitivity for the tuned RSE configuration is obtained as

$$S_{\text{RSE}} = \frac{h_{\text{SQL}}^2}{2} \left(\mathcal{K}_{\text{RSE}} + \frac{1}{\mathcal{K}_{\text{RSE}}} \right). \quad (3.4.20)$$

Here, $\mathcal{K}_{\text{RSE}}$ is the effective ponderomotive coupling in the RSE configuration, which has the same functional form as that of a Fabry–Pérot Michelson interferometer, but with an effective linewidth modified by the signal recycling mirror. It is defined as

$$\mathcal{K}_{\text{RSE}} \equiv -\frac{\mathcal{K}\tau^2}{1 + \rho^2 + 2\rho \cos 2\beta} \simeq \frac{8P\omega_0}{mcL} \frac{2\gamma_{\text{RSE}}}{\Omega^2(\gamma_{\text{RSE}}^2 + \Omega^2)}. \quad (3.4.21)$$

The effective linewidth including the signal recycling mirror is defined as

$$\gamma_{\text{RSE}} \equiv \frac{1 + \rho}{1 - \rho} \gamma. \quad (3.4.22)$$

Figure 3.7 shows the sensitivity curve for the tuned RSE configuration. Compared to the FPMI configuration, the addition of the signal recycling mirror broadens the sensitivity bandwidth. This configuration is employed in current detectors such as Advanced LIGO.

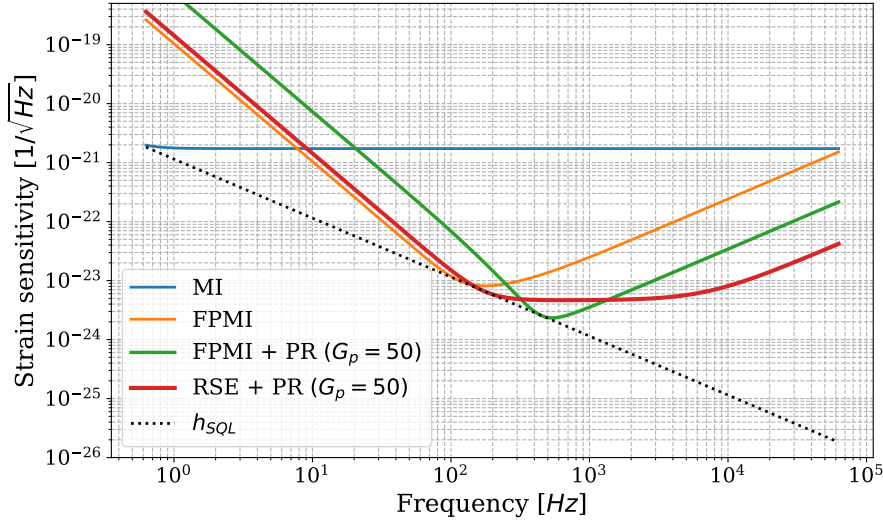


Figure 3.7: Strain sensitivity of the RSE configuration compared with that of a Fabry–Pérot Michelson interferometer.

3.4.4 Detuned Resonant Sideband Extraction

More generally, by slightly detuning the signal extraction cavity (SEC) from the carrier resonance, it is possible to realize an interferometer configuration with enhanced sensitivity in a specific frequency band. This configuration is referred to as detuned resonant sideband extraction (detuned RSE). In the detuned RSE configuration, the effective linewidth γ_{RSE}^d is given by

$$\gamma_{\text{RSE}}^d = \frac{\tau^2}{1 + \rho^2 + 2\rho \cos 2\beta} \gamma, \quad (3.4.23)$$

where ρ and τ are the amplitude reflectivity and transmissivity of the signal recycling mirror, respectively. From Eq. (3.4.13), the strain sensitivity of the detuned RSE configuration is obtained as

$$S_{\text{RSE}}^d = \frac{h_{\text{SQL}}^2}{2\mathcal{K}_{\text{RSE}}} \frac{|C_{11} \sin \zeta + C_{12} \cos \zeta|^2 + |C_{21} \sin \zeta + C_{22} \cos \zeta|^2}{\tau^2 |D_1 \sin \zeta + D_2 \cos \zeta|^2}, \quad (3.4.24)$$

where ζ is the homodyne angle, which specifies the quadrature direction to be measured at the output.

Figure 3.8 shows the sensitivity curve for the detuned RSE configuration. Due to the detuning of the SEC, two characteristic dips appear in the sensitivity. The dip at higher

frequencies arises from the resonant enhancement of the gravitational-wave sidebands in the optical cavity, which increases the signal response. In contrast, the dip at lower frequencies originates from optomechanical coupling between radiation pressure and the test-mass motion.

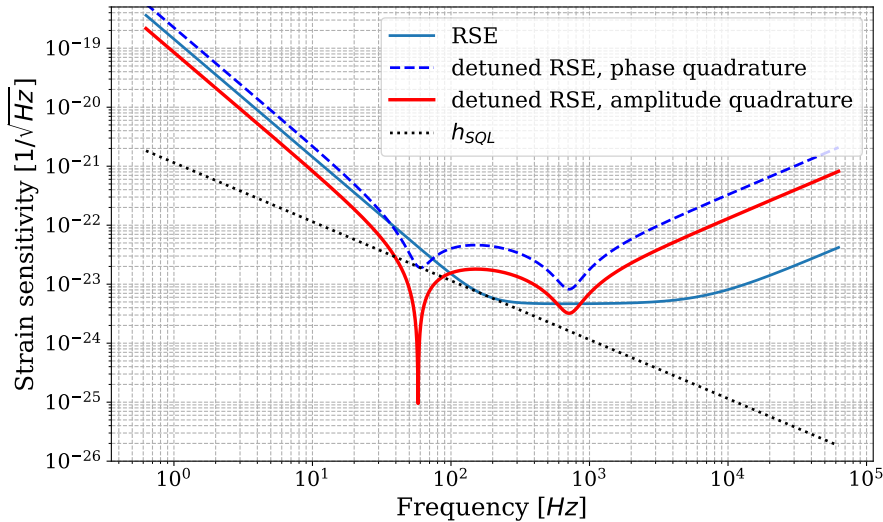


Figure 3.8: Sensitivity of the detuned RSE configuration. The SEC detuning is set to $\phi = \pi/2 - 0.75$ rad. The blue dashed curve shows the sensitivity obtained by measuring the phase quadrature, while the red solid curve corresponds to the amplitude quadrature.

Experimental demonstrations of the detuned RSE configuration have already been performed. Furthermore, detuned RSE is planned to be employed as the low-frequency-optimized configuration of the next-generation gravitational-wave detector, the Einstein Telescope.

Symbol	Description	Value
$\hbar$	Reduced Planck constant	$1.0545718 \times 10^{-34}$ J s
m	Test mass	40 kg
L	Arm length	4000 m
c	Speed of light	3.0×10^8 m/s
ω_0	Laser angular frequency	$2\pi c/1064$ nm
P_{in}	Input power	56 W
G_p	Power recycling gain	50
G_{arm}	Arm cavity gain	260
P_c	Circulating power	364 kW
T_i	Input mirror transmissivity	0.014
R_s	Signal recycling mirror reflectivity	0.8

Table 3.1: Parameters used in the interferometer model. The values are based on those of the LIGO Hanford detector during the O4a observing run [2].

3.5 Frequency-Dependent Squeezing

In this section, we discuss frequency-dependent squeezing, a technique that enables broadband reduction of quantum noise beyond the standard quantum limit (SQL). As discussed in the previous sections, quantum noise in laser interferometers originates from vacuum fluctuations entering through the dark port and manifests as radiation-pressure noise at low frequencies and shot noise at high frequencies. To overcome this quantum-noise trade-off, Kimble *et al.* [23] introduced the concept of *frequency-dependent squeezing*, in which the squeezing angle of the injected squeezed state is optimally rotated as a function of frequency to improve sensitivity over a broad band. When a squeezed state is injected into the interferometer, the resulting strain sensitivity is given by

$$S_{\text{sqz}} = \frac{h_{\text{SQL}}^2}{2} \left(\frac{1}{\mathcal{K}_{\text{RSE}}} + \mathcal{K}_{\text{RSE}} \right) [\cosh 2r - \cos(2(\Phi - \lambda_{\text{sq}})) \sinh 2r], \quad (3.5.1)$$

where $\Phi = \text{arccot}(\mathcal{K}_{\text{RSE}})$, λ_{sq} is the squeezing angle, and r is the squeezing factor. Figure 3.9 shows the strain sensitivity of an RSE interferometer with squeezing injection.

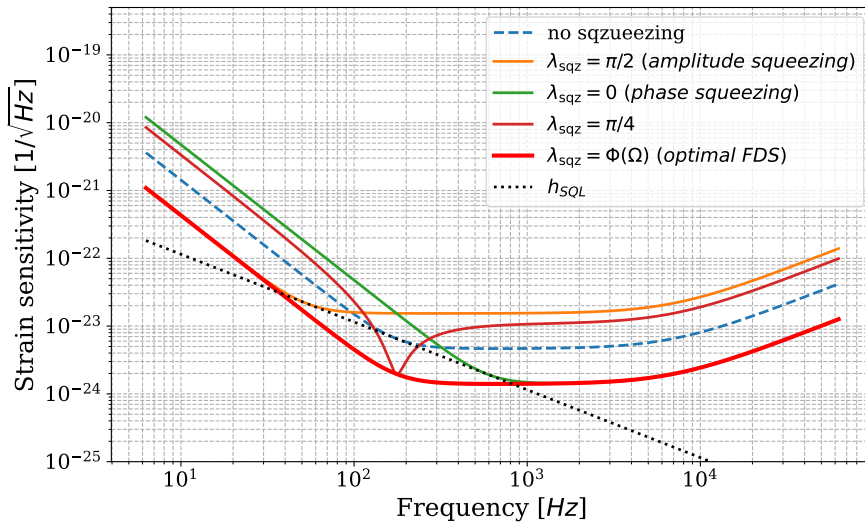


Figure 3.9: Strain sensitivity with frequency-independent squeezing and frequency-dependent squeezing.

For $\lambda_{\text{sq}} = \pi/2$, the strain sensitivity reduces to

$$S_{\text{sqz}}^{\text{phase}} = \frac{h_{\text{SQL}}^2}{2} \left(\frac{e^{-2r}}{\mathcal{K}_{\text{RSE}}} + \mathcal{K}_{\text{RSE}} e^{2r} \right), \quad (3.5.2)$$

which corresponds to *phase squeezing*. In this case, shot noise originating from the phase quadrature is reduced, leading to improved sensitivity at high frequencies. This noise-reduction scheme, in which the squeezing angle is independent of frequency, is referred to as *frequency-independent squeezing* (FIS). While phase squeezing improves the high-frequency sensitivity, the coupling induced by ponderomotive squeezing enhances the contribution of the amplitude quadrature fluctuation at low frequencies, thereby degrading the sensitivity.

The strength of this coupling is characterized by the ponderomotive factor $\mathcal{K}_{\text{RSE}}$, which depends on both the interferometer configuration and the frequency. Therefore, by choosing a squeezing angle that is optimized as a function of frequency, it is possible to improve the sensitivity over the entire frequency band. This technique is known as *frequency-dependent squeezing* (FDS).

The optimal frequency-dependent squeezing angle is given by

$$\lambda_{\text{sq}}^{\text{opt}}(\Omega) = \Phi(\Omega) = \text{arccot}(\mathcal{K}_{\text{RSE}}(\Omega)), \quad (3.5.3)$$

and the corresponding strain sensitivity becomes

$$S_{\text{sqz}}^{\text{opt}} = \frac{h_{\text{SQL}}^2}{2} \left(\frac{1}{\mathcal{K}_{\text{RSE}}} + \mathcal{K}_{\text{RSE}} \right) e^{-2r}. \quad (3.5.4)$$

Therefore, frequency-dependent squeezing allows broadband reduction of quantum noise, improving the sensitivity across all frequencies according to the squeezing level r .

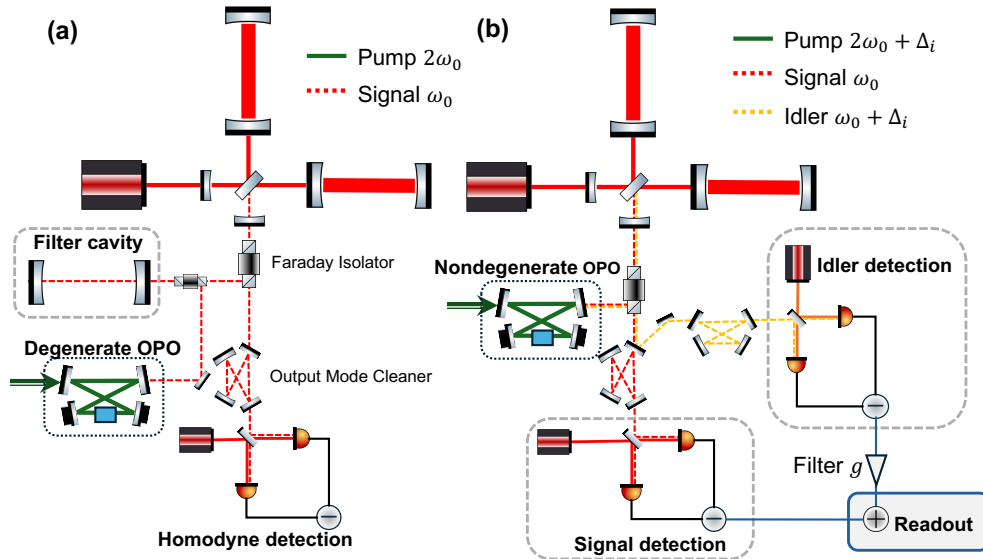


Figure 3.10: (a) Optical configuration for realizing frequency-dependent squeezing using a filter cavity. (b) Optical configuration based on the Einstein-Podolsky-Rosen (EPR) scheme.

3.5.1 Filter cavity

The most established method for introducing a frequency-dependent squeezing angle is the use of a filter cavity, originally proposed by Kimble *et al.* [23]. The optical configuration of this scheme is shown in Fig. 3.10(a). A filter cavity is a detuned Fabry-Pérot cavity that imparts a frequency-dependent quadrature rotation to the injected squeezed-vacuum field. As a result, amplitude squeezing is realized at low frequencies, while phase squeezing is obtained at high frequencies. In practice, frequency-dependent squeezing using filter cavities has been successfully demonstrated in Advanced LIGO and Virgo during the O4 observing run, achieving approximately 5 dB of quantum noise reduction [10]. The key design parameters of a filter cavity are the detuning δ from resonance and the cavity linewidth γ . The

detuning determines the center frequency at which the quadrature rotation occurs, while the linewidth controls the sharpness of the frequency dependence. To realize the narrow linewidth required for gravitational-wave interferometers, the mirrors forming the filter cavity must have high reflectivities. This leads to a high cavity finesse and, consequently, stringent requirements on optical losses inside the cavity. To relax these loss requirements, the filter cavity length must be increased so that the effect of optical loss per unit length is reduced. For this reason, kilometer-scale filter cavities are required for third-generation detectors such as the Einstein Telescope [24]. However, the construction and operation of such long, low-loss cavities pose significant technical challenges and introduce substantial economic costs, including the installation of additional vacuum infrastructure.

3.5.2 EPR squeezing

EPR squeezing is an alternative approach to realizing frequency-dependent squeezing by exploiting Einstein–Podolsky–Rosen (EPR) entanglement generated in a two-mode squeezed state². This scheme was proposed by Ma *et al.* [26], and its optical configuration is illustrated in Fig. 3.10(b). In this method, a two-mode squeezed-vacuum field is injected into the gravitational-wave interferometer. The signal field is frequency-matched to the interferometer carrier frequency, and its response includes both the gravitational-wave signal and quantum noise rotated by ponderomotive squeezing, as described by Eq. (3.4.13). In contrast, the idler field has a different carrier frequency³, and therefore experiences a different optical response. For the idler field, the interferometer effectively acts as a detuned cavity, providing a frequency-dependent quadrature rotation. After transmission through the interferometer, the signal and idler fields are spatially separated by an output mode cleaner (OMC) installed at the output port. Each field is then independently measured using homodyne detection. By appropriately combining the measurement results of the two fields, the phase rotation experienced by the idler field can be transferred to the signal field, thereby realizing frequency-dependent squeezing. Several tabletop experiments have successfully demonstrated this scheme in proof-of-principle setups [27–30]. While the EPR squeezing approach has the advantage that it does not require a long filter cavity, it suffers from an intrinsic drawback: because the measurement outcomes of two optical fields must be combined, the shot noise is effectively doubled. As discussed in Sec. 4.4, this results in a reduction of the effective squeezing level by approximately 3 dB compared to the filter-cavity-based approach.

3.5.3 Negative-mass spin oscillator

Another approach to frequency-dependent squeezing, based on quantum correlations with a negative-mass spin oscillator, was proposed by Khalili *et al.* [31]. This method employs quantum correlations between two-mode squeezed states generated by a nondegenerate optical parametric oscillator (OPO) and an atomic spin ensemble, such as ^{133}Cs . In this scheme, the signal field is injected into the gravitational-wave interferometer and

²The term EPR entanglement originates from the seminal discussion of quantum correlations by Einstein, Podolsky, and Rosen [25].

³The frequency separation between the signal and idler fields is determined by the free spectral range of the OPO and the pump frequency.

detected via homodyne readout at the output port. The idler field, on the other hand, interacts dispersively with the atomic spin ensemble and is measured using polarization homodyne detection. By combining the measurement results from the optical and atomic systems, the phase rotation experienced by the idler field is effectively transferred to the signal field, thus realizing frequency-dependent squeezing. The center frequency of the quadrature rotation is determined by the Larmor frequency of the atomic spin ensemble, which can be tuned by adjusting the external magnetic field. The sharpness of the frequency dependence is governed by the interaction strength between the atoms and light, characterized by the so-called readout rate. Recent tabletop experiments have demonstrated Larmor frequencies of approximately 10 kHz and readout rates of 4.1 kHz [32].

Chapter 4

Generating Squeezed States

This chapter describes the generation of squeezed states using nonlinear optical processes, methods for measuring quantum states of light, and the impact of optical losses on squeezed states. In Sec. 4.1, the response of an optical cavity is analyzed using the Langevin equation. Section 4.2 provides a brief overview of nonlinear optics, which is essential for the generation of squeezed states. In Sec. 4.3, the generation of squeezed states using a degenerate optical parametric oscillator (OPO) is discussed based on the Langevin formalism. Section 4.4 explains the generation of two-mode squeezed states using a nondegenerate OPO and shows how quantum entanglement reduces joint quadrature variances. In Sec. 4.5, optical detection schemes are introduced and the measurement of squeezed states is described. Finally, Sec. 4.6 discusses the effects of realistic factors, such as optical losses, on squeezed states.

4.1 Optical cavity equation of motion

Optical cavities store light through constructive interference and therefore play an essential role in gravitational-wave detectors, as discussed in Subsection 2.2.2. Beyond this application, optical cavities are widely used for mode separation in frequency and spatial degrees of freedom and for quadrature rotation in filter cavities. They also enable the efficient generation of squeezed states in optical parametric oscillators by enhancing the effective nonlinear interaction.

4.1.1 Cavity classical response

In order to establish a theoretical framework for later discussions, we first analyze the response of an optical cavity to external driving fields. The intracavity optical field is modeled as a harmonic oscillator, and its equations of motion are derived to describe the dynamical response of the cavity. This approach allows us to treat both the classical response and its extension to quantum fluctuations within a unified framework.

We consider an optical cavity consisting of four mirrors, arranged in a bow-tie configuration, as illustrated in Fig. 4.1. The annihilation operator of the intracavity field is denoted by $\hat{a}$, and the external fields incident on the cavity through the three ports¹ are represented by $\hat{A}_{\text{in}}$, $\hat{A}_{\text{out}}$, and $\hat{A}_{l,\text{in}}$. The operator $\hat{a}$ has units of $\sqrt{\text{photon}}$, while the

¹For clarity, one of the ports of the bow-tie cavity is omitted in the schematic and in the following analysis. This simplification does not affect the underlying physics, and the omitted port can be treated as an effective loss port.

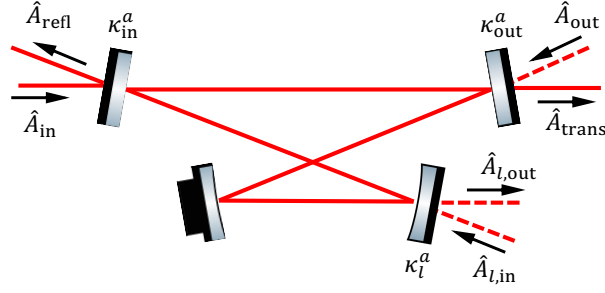


Figure 4.1: Schematic of an optical cavity with three ports. The reflected and transmitted power and phase responses depend on the coupling condition. In the over-coupled regime, most of the incident light is reflected near resonance, whereas in the critically coupled case, all incident light is transmitted. At resonance, the reflected field exhibits a π phase flip.

flux fields $\hat{A}_j$ have units of $\sqrt{\text{photon/s}}$. The decay rate κ_j^a associated with each port is determined by the mirror reflectivity R_j and the cavity round-trip storage time $\tau_c = L/c$ as

$$\kappa_j^a = -\frac{1}{\tau_c} \ln R_j, \quad (j = \text{in}, \text{out}, l). \quad (4.1.1)$$

The total decay rate of the cavity is given by the sum of the individual decay rates,

$$\kappa_a = \kappa_{in}^a + \kappa_{out}^a + \kappa_l^a. \quad (4.1.2)$$

The fields emitted from the cavity are denoted by $\hat{A}_{\text{refl}}$, $\hat{A}_{\text{trans}}$, and $\hat{A}_{l,\text{out}}$. The total Hamiltonian of the cavity system can be written as

$$H_{\text{tot}} = H_{\text{cav}} + H_{\text{bath}} + H_{\text{ext}}, \quad (4.1.3)$$

where the individual terms are defined as follows.

- The Hamiltonian of the intracavity field:

$$H_{\text{cav}} = \hbar\omega_0 \hat{a}^\dagger \hat{a}. \quad (4.1.4)$$

- The Hamiltonian of the external bath modes:

$$H_{\text{bath}} = \sum_j \int_{-\infty}^{+\infty} \hbar\omega \hat{A}_j^\dagger(\omega) \hat{A}_j(\omega) d\omega. \quad (4.1.5)$$

- The interaction Hamiltonian between the cavity field and the bath:

$$H_{\text{ext}} = i\hbar \sum_j \int_{-\infty}^{+\infty} \sqrt{2\kappa_j^a} \left(\hat{A}_j^\dagger(\omega) \hat{a} - \hat{a}^\dagger \hat{A}_j(\omega) \right) d\omega. \quad (4.1.6)$$

From the total Hamiltonian, the Langevin equation for the intracavity field $\hat{a}$ is obtained as

$$\dot{\hat{a}} = -\frac{i}{\hbar} [\hat{a}, H_{\text{cav}}] - \kappa_a \hat{a} + \sqrt{2\kappa_{in}^a} \hat{A}_{in} + \sqrt{2\kappa_{out}^a} \hat{A}_{out} + \sqrt{2\kappa_l^a} \hat{A}_{l,in}. \quad (4.1.7)$$

The first term represents the unitary time evolution of the closed cavity system; the second term describes the dissipation of the intracavity field, and the remaining terms account for the driving by external fields. Transforming into a rotating frame at the laser frequency ω_0 by $\hat{a}(t) \rightarrow \hat{a}(t)e^{i\omega_0 t}$, the equations of motion become

$$\dot{\hat{a}} = -(\kappa_a - i\Delta_{\text{cav}}^a)\hat{a} + \sqrt{2\kappa_{\text{in}}^a}\hat{A}_{\text{in}} + \sqrt{2\kappa_{\text{out}}^a}\hat{A}_{\text{out}} + \sqrt{2\kappa_l^a}\hat{A}_{l,\text{in}}, \quad (4.1.8)$$

$$\dot{\hat{a}}^\dagger = -(\kappa_a + i\Delta_{\text{cav}}^a)\hat{a}^\dagger + \sqrt{2\kappa_{\text{in}}^a}\hat{A}_{\text{in}}^\dagger + \sqrt{2\kappa_{\text{out}}^a}\hat{A}_{\text{out}}^\dagger + \sqrt{2\kappa_l^a}\hat{A}_{l,\text{in}}^\dagger. \quad (4.1.9)$$

Here, $\Delta_{\text{cav}}^a = \omega_0 - \omega_{\text{cav}}^a$ denotes the detuning between the laser angular frequency and the cavity resonance. Taking the Fourier transform of the equations of motion, we obtain

$$i\omega \tilde{\mathbf{a}}(\omega) = \mathbf{M}_{\text{cav}} \tilde{\mathbf{a}} + \sum_j \sqrt{2\kappa_j^a} \tilde{\mathbf{A}}_j(\omega), \quad (4.1.10)$$

where we define the matrix $\mathbf{M}_{\text{cav}}$ and the vectors $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{A}}_j$ as

$$\mathbf{M}_{\text{cav}} = \begin{bmatrix} -\kappa_a + i\Delta_{\text{cav}}^a & 0 \\ 0 & -\kappa_a - i\Delta_{\text{cav}}^a \end{bmatrix}, \quad (4.1.11)$$

$$\tilde{\mathbf{a}}(\omega) = \begin{bmatrix} \tilde{a}(\omega) \\ \tilde{a}^\dagger(-\omega) \end{bmatrix}, \quad \tilde{\mathbf{A}}_j(\omega) = \begin{bmatrix} \tilde{A}_j(\omega) \\ \tilde{A}_j^\dagger(-\omega) \end{bmatrix}. \quad (4.1.12)$$

In the rotating frame, the Fourier frequency ω represents the angular frequency of sidebands relative to the carrier and is denoted by Ω in the previous sections. Solving Eq. (4.1.10) for $\tilde{\mathbf{a}}(\omega)$ yields

$$\tilde{\mathbf{a}} = \sum_j \sqrt{2\kappa_j^a} (i\omega - \mathbf{M}_{\text{cav}})^{-1} \tilde{\mathbf{A}}_j. \quad (4.1.13)$$

To obtain the classical transfer functions of the cavity, we impose boundary conditions relating the intracavity field to the input and output fields at each port. The boundary conditions are given by

$$\begin{aligned} \tilde{A}_{\text{in}} + \tilde{A}_{\text{refl}} &= \sqrt{2\kappa_{\text{in}}^a} \tilde{a}, \\ \tilde{A}_{\text{out}} + \tilde{A}_{\text{trans}} &= \sqrt{2\kappa_{\text{out}}^a} \tilde{a}, \\ \tilde{A}_{l,\text{in}} + \tilde{A}_{l,\text{out}} &= \sqrt{2\kappa_l^a} \tilde{a}. \end{aligned} \quad (4.1.14)$$

Using these boundary conditions together with Eq. (4.1.13), the reflected and transmitted fields can be expressed in terms of the incident fields at each port:

$$\tilde{A}_{\text{refl}} = \frac{(2\kappa_{\text{in}}^a - \kappa_a - i(\omega - \Delta_{\text{cav}}^a)) \tilde{A}_{\text{in}} + 2\sqrt{\kappa_{\text{in}}^a \kappa_{\text{out}}^a} \tilde{A}_{\text{out}} + 2\sqrt{\kappa_{\text{in}}^a \kappa_l^a} \tilde{A}_{l,\text{in}}}{\kappa_a + i(\omega - \Delta_{\text{cav}}^a)}, \quad (4.1.15)$$

$$\tilde{A}_{\text{trans}} = \frac{2\sqrt{\kappa_{\text{in}}^a \kappa_{\text{out}}^a} \tilde{A}_{\text{in}} + (2\kappa_{\text{out}}^a - \kappa_a - i(\omega - \Delta_{\text{cav}}^a)) \tilde{A}_{\text{out}} + 2\sqrt{\kappa_{\text{out}}^a \kappa_l^a} \tilde{A}_{l,\text{in}}}{\kappa_a + i(\omega - \Delta_{\text{cav}}^a)}. \quad (4.1.16)$$

For simplicity, we consider the case where only the input port is driven, i.e., $\tilde{A}_{\text{out}} = \tilde{A}_{l,\text{in}} = 0$. The classical transfer functions for the reflected and transmitted fields are then

$$\frac{\tilde{A}_{\text{refl}}}{\tilde{A}_{\text{in}}} = \frac{2\kappa_{\text{in}}^a - \kappa_a - i(\omega - \Delta_{\text{cav}}^a)}{\kappa_a + i(\omega - \Delta_{\text{cav}}^a)}, \quad (4.1.17)$$

$$\frac{\tilde{A}_{\text{trans}}}{\tilde{A}_{\text{in}}} = \frac{2\sqrt{\kappa_{\text{in}}^a \kappa_{\text{out}}^a}}{\kappa_a + i(\omega - \Delta_{\text{cav}}^a)}. \quad (4.1.18)$$

The cavity response depends on the relative magnitudes of the decay rates associated with each port. The cavity can be classified into three coupling regimes. If the input coupling is weaker than the sum of the other decay channels, $\kappa_{\text{in}}^a < \kappa_{\text{out}}^a + \kappa_l^a$, the cavity is said to be under-coupled. When $\kappa_{\text{in}}^a = \kappa_{\text{out}}^a + \kappa_l^a$, the cavity is critically coupled. If the input coupling dominates, $\kappa_{\text{in}}^a > \kappa_{\text{out}}^a + \kappa_l^a$, the cavity is over-coupled. Figure 4.2 shows the power and phase responses of the reflected and transmitted fields near resonance.

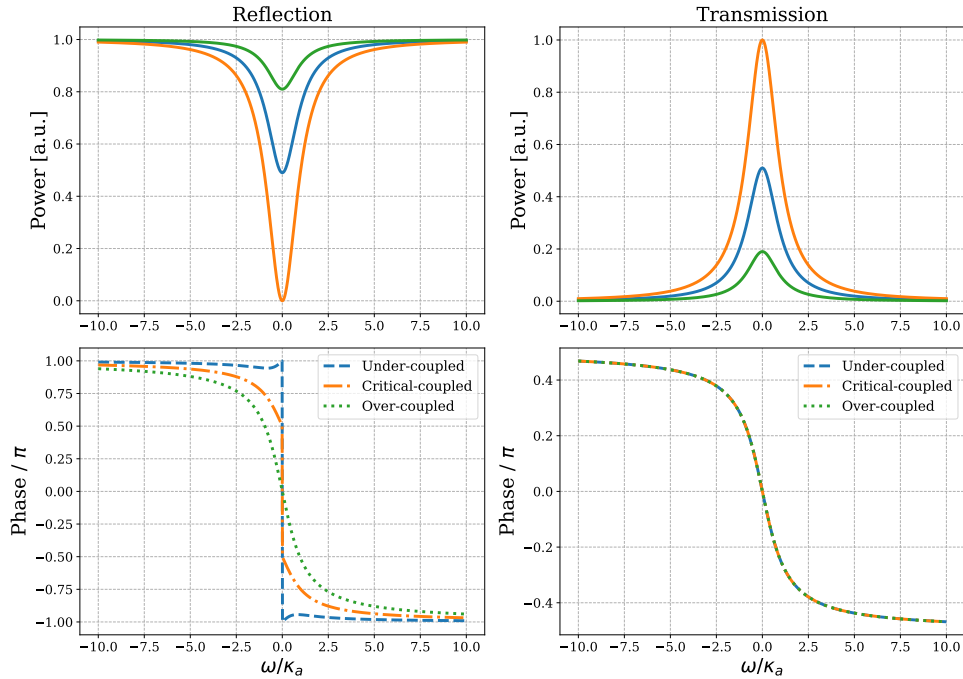


Figure 4.2: Power and phase responses of the reflected and transmitted fields around cavity resonance. In the over-coupled regime, most of the incident light is reflected near resonance, whereas under critical coupling all the incident light is transmitted. The reflected field exhibits a π phase flip at resonance.

4.1.2 Cavity noise response

To describe the response of an optical cavity to fluctuations, the intracavity field and the input fields are linearized around their steady-state values. Each field operator is decomposed into a steady classical component and a fluctuation term as

$$\tilde{A}_j = \bar{A}_j + \delta\tilde{A}_j, \quad (4.1.19)$$

and similarly for the annihilation and creation operators,

$$\tilde{a} = \bar{a} + \delta\tilde{a}, \quad (4.1.20)$$

$$\tilde{a}^\dagger = \bar{a}^\dagger + \delta\tilde{a}^\dagger. \quad (4.1.21)$$

Substituting these expressions into the Langevin equation Eq. (4.1.7) and keeping only terms linear in the fluctuations, the fluctuation components of the reflected and transmitted fields can be obtained in a form analogous to Eq. (4.1.16):

$$\delta\tilde{A}_{\text{refl}} = \frac{(2\kappa_{\text{in}}^a - \kappa_a - i(\omega - \Delta_{\text{cav}}^a))\delta\tilde{A}_{\text{in}} + 2\sqrt{\kappa_{\text{in}}^a\kappa_{\text{out}}^a}\delta\tilde{A}_{\text{out}} + 2\sqrt{\kappa_{\text{in}}^a\kappa_l^a}\delta\tilde{A}_{l,\text{in}}}{\kappa_a + i(\omega - \Delta_{\text{cav}}^a)}, \quad (4.1.22)$$

$$\delta\tilde{A}_{\text{trans}} = \frac{2\sqrt{\kappa_{\text{in}}^a\kappa_{\text{out}}^a}\delta\tilde{A}_{\text{in}} + (2\kappa_{\text{out}}^a - \kappa_a - i(\omega - \Delta_{\text{cav}}^a))\delta\tilde{A}_{\text{out}} + 2\sqrt{\kappa_{\text{out}}^a\kappa_l^a}\delta\tilde{A}_{l,\text{in}}}{\kappa_a + i(\omega - \Delta_{\text{cav}}^a)}. \quad (4.1.23)$$

Next, we consider the response of the system in terms of quadrature operators. The transformation from the sideband basis $\delta\hat{\mathbf{A}}_j$ to the quadrature basis $\delta\hat{\mathbf{X}}^j$ is given by

$$\delta\hat{\mathbf{X}}^j = \mathbf{\Gamma}\delta\hat{\mathbf{A}}_j, \quad (4.1.24)$$

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}. \quad (4.1.25)$$

Using this transformation, the quadrature responses at each port are written as

$$\delta\tilde{\mathbf{X}}_{\text{cav}}^{\text{refl}} = \frac{(2\kappa_{\text{in}}^a - \kappa_a - i(\omega - \Delta_{\text{cav}}^a))\delta\tilde{\mathbf{X}}_{\text{in}} + 2\sqrt{\kappa_{\text{in}}^a\kappa_{\text{out}}^a}\delta\tilde{\mathbf{X}}_{\text{out}} + 2\sqrt{\kappa_{\text{in}}^a\kappa_l^a}\delta\tilde{\mathbf{X}}_{l,\text{in}}}{\kappa_a + i(\omega - \Delta_{\text{cav}}^a)}, \quad (4.1.26)$$

$$\delta\tilde{\mathbf{X}}_{\text{cav}}^{\text{trans}} = \frac{2\sqrt{\kappa_{\text{in}}^a\kappa_{\text{out}}^a}\delta\tilde{\mathbf{X}}_{\text{in}} + (2\kappa_{\text{out}}^a - \kappa_a - i(\omega - \Delta_{\text{cav}}^a))\delta\tilde{\mathbf{X}}_{\text{out}} + 2\sqrt{\kappa_{\text{out}}^a\kappa_l^a}\delta\tilde{\mathbf{X}}_{l,\text{in}}}{\kappa_a + i(\omega - \Delta_{\text{cav}}^a)}. \quad (4.1.27)$$

In the absence of optical loss ($\delta\tilde{\mathbf{X}}_{l,\text{in}} = 0$), the quadrature variances of the reflected and transmitted fields are given by²:

$$V_{\text{cav}}^{\text{refl}} = 1 + \frac{(2\kappa_{\text{in}}^a - \kappa_a)^2 + \omega^2}{\kappa_a^2 + \omega^2} (V_{\text{in}} - 1), \quad (4.1.28)$$

$$V_{\text{cav}}^{\text{trans}} = 1 + \frac{4\kappa_{\text{in}}^a\kappa_{\text{out}}^a}{\kappa_a^2 + \omega^2} (V_{\text{in}} - 1). \quad (4.1.29)$$

The variances of the reflected and transmitted fields are shown in Fig. 4.3. From this result, several features can be understood. For an input state with quadrature variance $V_{\text{in}} = 10$, the variance of the transmitted field is preserved near the cavity resonance, whereas it is suppressed outside the cavity linewidth. Thus, the cavity acts as a low-pass

²Here we use the relations $\langle\delta\tilde{\mathbf{X}}_{\text{in}}\delta\tilde{\mathbf{X}}_{\text{in}}^\dagger\rangle = V_{\text{in}}$ and $\langle\delta\tilde{\mathbf{X}}_{\text{in}}\delta\tilde{\mathbf{X}}_{\text{out}}^\dagger\rangle = 0$.

filter for the quadrature variance of the transmitted field, while it behaves as a high-pass filter for the variance of the reflected field.

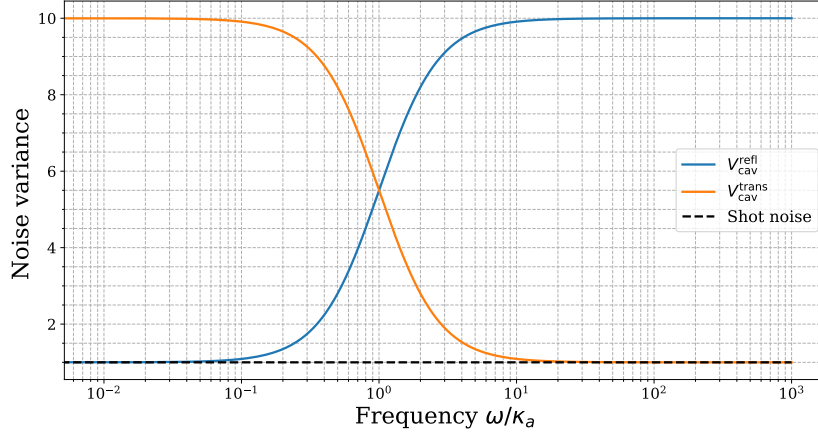


Figure 4.3: Quadrature variances of the transmitted and reflected fields from the cavity.

4.2 Nonlinear Optical Interactions

Nonlinear optical interactions arise when a dielectric medium is irradiated by a strong electromagnetic field, leading to a nonlinear response of the medium. In general, when an electric field is incident on a dielectric material, it induces an electric polarization. For weak fields, the polarization oscillates linearly in response to the applied field. However, under sufficiently strong electric fields, the linear approximation breaks down and nonlinear optical effects become significant. The relation between the polarization and the electric field in a nonlinear medium can be expressed as

$$\begin{aligned} \mathbf{P} &= \varepsilon_0 \chi^{(1)} \mathbf{E} + \varepsilon_0 \chi^{(2)} \mathbf{E} \mathbf{E} + \varepsilon_0 \chi^{(3)} \mathbf{E} \mathbf{E} \mathbf{E} + \dots \\ &\equiv \mathbf{P}_L + \mathbf{P}_{NL}, \end{aligned} \quad (4.2.1)$$

where ε_0 and $\chi^{(1)}$ denote the vacuum permittivity and the linear susceptibility, respectively. The n -th order nonlinear susceptibility $\chi^{(n)}$ is a tensor of rank $n+1$. The total polarization is decomposed into the linear part $\mathbf{P}_L$ and the nonlinear part $\mathbf{P}_{NL}$.

The interaction between the electric field and the nonlinear polarization in a medium is described by Maxwell's equations, which lead to the following wave equation:

$$\nabla^2 \mathbf{E} - \frac{1}{c_n^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2}, \quad (4.2.2)$$

where the refractive index of the medium is given by $n = \sqrt{\varepsilon/\varepsilon_0} = \sqrt{1 + \chi^{(1)}}$, and $c_n = c/n$. This equation indicates that the nonlinear polarization $\mathbf{P}_{NL}$ acts as a source term for the electromagnetic field.

In this study, we focus exclusively on second-order nonlinear optical processes. Representative second-order nonlinear interactions include the following processes³. Schematic illustrations of these processes are shown in Fig. 4.4.

- **Difference-frequency generation** — A single photon with frequency ω_1 is converted into two photons with frequencies ω_2 and ω_3 inside a nonlinear medium. Energy conservation requires

$$\omega_1 = \omega_2 + \omega_3, \quad (4.2.3)$$

while momentum conservation imposes

$$\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3. \quad (4.2.4)$$

When this process is used to amplify a specific frequency component, it is referred to as optical parametric amplification (OPA).

- **Sum-frequency generation** — Two photons with frequencies ω_1 and ω_2 are converted into a single photon with frequency ω_3 . The process satisfies the energy conservation law

$$\omega_1 + \omega_2 = \omega_3, \quad (4.2.5)$$

and the momentum conservation condition

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3, \quad (4.2.6)$$

where $\mathbf{k}_i$ denotes the wave vector. In the special case $\omega_1 = \omega_2$, the relation $2\omega_1 = \omega_3$ holds, and this process is known as second-harmonic generation (SHG). In this work, SHG is employed to generate the pump field required for squeezed-state generation.

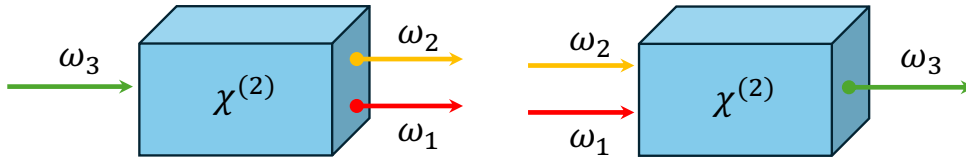


Figure 4.4: Schematic diagrams of difference-frequency generation (left) and sum-frequency generation (right). The blue cube represents a nonlinear medium.

4.2.1 Phase Matching

The direction and efficiency of energy transfer among different frequency components generated by nonlinear optical interactions are governed by the phase-matching condition. As a concrete example, we consider second-harmonic generation (SHG). A fundamental wave with angular frequency ω_1 propagates through a $\chi^{(2)}$ nonlinear medium of length L , generating a wave at frequency $\omega_3 = 2\omega_1$. The electric fields are expressed using slowly varying envelopes as

$$E_1(z, t) = A_1(z)e^{i(\omega_1 t - k_1 z)}, \quad E_3(z, t) = A_3(z)e^{i(2\omega_1 t - k_3 z)}, \quad (4.2.7)$$

³Typical third-order nonlinear effects include the optical Kerr effect.

where $A_j(z)$ denotes the slowly varying complex envelope. The phase mismatch is defined as

$$\Delta k = k_3 - 2k_1. \quad (4.2.8)$$

Substituting the second-order nonlinear polarization $P^{(2)} = \varepsilon_0 \chi^{(2)} E^2$ into the wave equation and applying the slowly varying envelope approximation (SVEA)⁴, the coupled-wave equations for SHG are obtained as

$$\frac{dA_3}{dz} = -i \frac{\omega_1 \chi^{(2)}}{n_3 c} A_1^2 e^{-i\Delta k z}, \quad (4.2.9)$$

$$\frac{dA_1}{dz} = -i \frac{\omega_1 \chi^{(2)}}{2n_1 c} A_1^* A_3 e^{i\Delta k z}, \quad (4.2.10)$$

where n_1 and n_3 denote the refractive indices at frequencies ω_1 and ω_3 , respectively. Assuming that the fundamental wave is sufficiently strong and remains undepleted, $A_1(z) \simeq A_0$, the equation for A_3 reduces to

$$\frac{dA_3}{dz} = -i \frac{\omega_1 \chi^{(2)}}{n_3 c} A_0^2 e^{-i\Delta k z}. \quad (4.2.11)$$

Integrating this equation with the initial condition $A_3(0) = 0$, we obtain

$$A_3(z) = \frac{\omega_1 \chi^{(2)}}{n_3 c} \frac{A_0^2}{\Delta k} \left(1 - e^{-i\Delta k z}\right). \quad (4.2.12)$$

The resulting second-harmonic electric field is given by

$$E_3(z, t) = \frac{\omega_1 \chi^{(2)}}{n_3 c} \frac{A_0^2}{\Delta k} \left(1 - e^{-i\Delta k z}\right) e^{i(2\omega_1 t - k_3 z)}. \quad (4.2.13)$$

Consequently, the intensity of the generated second-harmonic wave after propagating through the medium is

$$I_{2\omega}(L) \propto L^2 \text{sinc}^2\left(\frac{\Delta k L}{2}\right), \quad (4.2.14)$$

demonstrating that the conversion efficiency strongly depends on the phase mismatch Δk . When $\Delta k = 0$, the phase-matching condition is satisfied, and the generated second-harmonic waves add constructively throughout the medium, yielding maximum conversion efficiency.

In general, however, material dispersion causes the refractive indices at the fundamental and second-harmonic frequencies to differ, leading to

$$\Delta k = \frac{n_3 \omega_3}{c} - \frac{2n_1 \omega_1}{c} \neq 0. \quad (4.2.15)$$

As a result, the generated fields gradually dephase inside the medium, and destructive

⁴The SVEA assumes that the envelope varies slowly along the propagation direction, such that

$$\left| \frac{\partial^2 A_j}{\partial z^2} \right| \ll \left| 2k_j \frac{\partial A_j}{\partial z} \right|,$$

allowing the second-order derivative term to be neglected.

interference limits the growth of the second-harmonic intensity for long interaction lengths. To overcome this limitation, phase matching techniques such as birefringent phase matching and quasi-phase matching are widely employed. In practice, second-order nonlinear optical processes are realized using nonlinear crystals.

4.2.2 Birefringent Phase Matching

Phase matching based on birefringence can be classified into two types: Type I and Type II phase matching. In Type I phase matching, the two fundamental waves have the same polarization, while the generated second-harmonic wave is polarized orthogonally to the fundamental waves. In contrast, in Type II phase matching, the two fundamental waves have orthogonal polarizations, and the second-harmonic wave is polarized along one of them. Phase matching among the three interacting waves is achieved by finely tuning the refractive indices through adjustment of the crystal orientation with respect to the incident beam and by controlling the crystal temperature. These degrees of freedom allow compensation for material dispersion and enable efficient nonlinear frequency conversion.

4.2.3 Quasi-Phase Matching

Quasi-phase matching (QPM) is a phase-matching technique realized by periodically reversing the sign of the nonlinear susceptibility inside a crystal. When phase mismatch $\Delta k \neq 0$ exists due to refractive index dispersion, the generated nonlinear polarization gradually accumulates a phase difference relative to the fundamental wave as it propagates through the medium. Beyond a certain propagation length, destructive interference suppresses the frequency conversion efficiency. The characteristic length over which a phase difference of π accumulates between the fundamental and generated waves is referred to as the coherence length, defined as

$$l_c = \frac{\pi}{|\Delta k|}. \quad (4.2.16)$$

In QPM, the polarization direction of the crystal is periodically inverted with a period equal to the coherence length l_c . This periodic poling effectively resets the phase of the generated field, enabling constructive interference to be maintained throughout the crystal and resulting in a monotonic increase in conversion efficiency. Figure 4.5 illustrates the relationship between the domain-inverted structure and the growth of the generated field intensity. Unlike birefringent phase matching, QPM does not impose polarization constraints arising from refractive index anisotropy. As a result, greater freedom in polarization selection relative to the crystal axes is available, allowing access to the largest nonlinear tensor components of the material. Representative periodically poled nonlinear materials include periodically poled lithium niobate (PPLN), periodically poled lithium tantalate (PPLT), and periodically poled potassium titanyl phosphate (PPKTP).

4.3 Degenerate OPO equation of motion

Squeezed states can be generated by operating an optical parametric oscillator (OPO) below threshold, where a nonlinear crystal is placed inside an optical cavity. In an OPO,

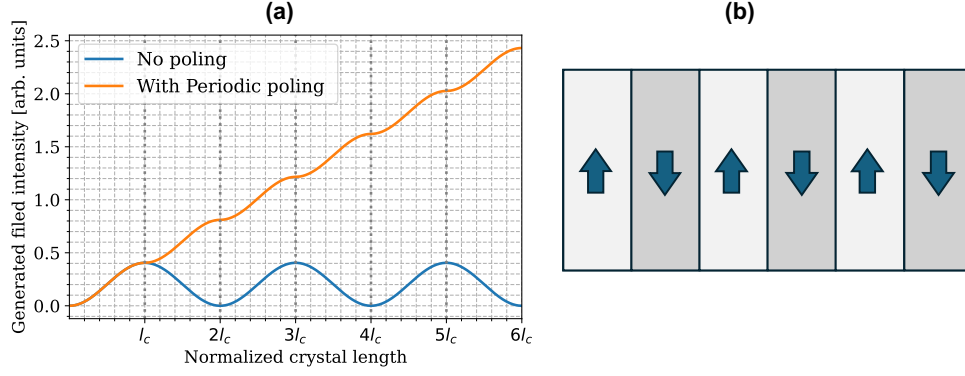


Figure 4.5: (a) Schematic illustration of the generated field intensity along the crystal length. Without phase matching, the intensity is suppressed by destructive interference, whereas periodic domain inversion enables monotonic growth. (b) Domain pattern of a quasi-phase-matched crystal.

the pump field interacts with the nonlinear medium and undergoes parametric down-conversion, producing a signal and an idler field. In this section, we consider the degenerate case, in which the signal and idler frequencies coincide. We formulate the cavity dynamics including the nonlinear interaction by introducing the Hamiltonian of the OPO cavity system, and then derive the Langevin equations. Using these equations, we evaluate the classical parametric gain and the noise variances of the squeezed vacuum state. A schematic overview of the OPO setup is shown in Fig. 4.6. The total Hamiltonian of the cavity system

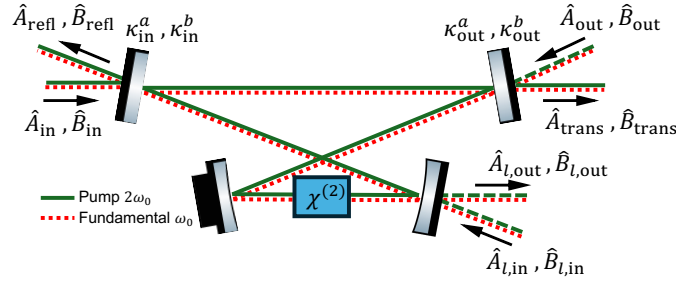


Figure 4.6: Schematic of an optical parametric oscillator (OPO) cavity. A nonlinear crystal inside the cavity enables parametric interaction between the fundamental field at ω_0 and the pump field at $2\omega_0$, generating squeezed states below threshold.

containing an OPO is written as

$$H_{\text{tot}}^{\text{OPO}} = H_{\text{cav}}^{\text{OPO}} + H_{\text{bath}}^{\text{OPO}} + H_{\text{ext}}^{\text{OPO}}, \quad (4.3.1)$$

where the three terms represent the intracavity Hamiltonian, the bath Hamiltonian, and the interaction between the cavity fields and the external bath modes, respectively.

- $H_{\text{cav}}^{\text{OPO}}$ represents the closed intracavity system:

$$H_{\text{cav}}^{\text{OPO}} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + 2\hbar\omega_0 \hat{b}^\dagger \hat{b} + \frac{i\hbar}{2} \left(\epsilon \hat{a}^{\dagger 2} \hat{b} - \epsilon^* \hat{a}^2 \hat{b}^\dagger \right). \quad (4.3.2)$$

- $H_{\text{bath}}^{\text{OPO}}$ represents the external continuum modes:

$$H_{\text{bath}}^{\text{OPO}} = \sum_j \int_{-\infty}^{+\infty} \hbar \omega \left(\hat{A}_j^\dagger(\omega) \hat{A}_j(\omega) + \hat{B}_j^\dagger(\omega) \hat{B}_j(\omega) \right) d\omega. \quad (4.3.3)$$

- $H_{\text{ext}}^{\text{OPO}}$ describes the coupling between the cavity fields and the bath:

$$H_{\text{ext}}^{\text{OPO}} = i\hbar \sum_j \int_{-\infty}^{+\infty} \sqrt{2\kappa_j^a} \left(\hat{A}_j^\dagger(\omega) \hat{a} - \hat{a}^\dagger \hat{A}_j(\omega) \right) d\omega. \quad (4.3.4)$$

Here, $\hat{a}$ and $\hat{b}$ denote the intracavity fundamental field at ω_0 and the second-harmonic (pump) field at $2\omega_0$, respectively. The operators $\hat{A}_j$ and $\hat{B}_j$ represent the corresponding external flux fields, κ_j^a denotes the decay rate of the fundamental field through each port, and ϵ is the nonlinear coupling parameter.

From the Hamiltonian in Eq. (4.3.1), the Langevin equations for the intracavity fields are obtained as

$$\dot{\hat{a}} = -(\kappa_a - i\Delta_{\text{cav}}^a) \hat{a} + \epsilon \hat{b} \hat{a}^\dagger + \sqrt{2\kappa_{\text{in}}^a} \hat{A}_{\text{in}} + \sqrt{2\kappa_{\text{out}}^a} \hat{A}_{\text{out}} + \sqrt{2\kappa_l^a} \hat{A}_{l,\text{in}}, \quad (4.3.5)$$

$$\dot{\hat{b}} = -(\kappa_b - i\Delta_{\text{cav}}^b) \hat{b} - \frac{\epsilon^*}{2} \hat{a}^2 + \sqrt{2\kappa_{\text{in}}^b} \hat{B}_{\text{in}} + \sqrt{2\kappa_{\text{out}}^b} \hat{B}_{\text{out}} + \sqrt{2\kappa_l^b} \hat{B}_{l,\text{in}}. \quad (4.3.6)$$

In the following, we analyze the OPO response by solving these equations in the frequency domain.

4.3.1 Classical response of a degenerate OPO

We first discuss the classical response of an OPO operated below threshold. To analyze the pump field that drives the parametric interaction, we impose the following conditions:

- The cavity is resonant for the pump field ($\Delta_{\text{cav}}^b = 0$).
- The pump field is in a steady state ($\dot{\hat{b}} = 0$).
- The pump field is sufficiently strong such that depletion due to the nonlinear interaction can be neglected (i.e., the back-action term proportional to $\hat{a}^2$ is ignored).

When the OPO cavity is resonant for both the fundamental and pump fields ($\Delta_{\text{cav}}^a = \Delta_{\text{cav}}^b = 0$), Eq. (4.3.6) yields the intracavity pump field

$$\hat{b} = \frac{\sqrt{2\kappa_{\text{in}}^b} \hat{B}_{\text{in}} + \sqrt{2\kappa_{\text{out}}^b} \hat{B}_{\text{out}} + \sqrt{2\kappa_l^b} \hat{B}_{l,\text{in}}}{\kappa_b}. \quad (4.3.7)$$

This expression shows that the intracavity pump amplitude is determined by the incident pump fields and the decay rates of the cavity ports.

To derive the nonlinear gain of the fundamental flux field induced by the pump field, we parameterize the nonlinear coupling term by introducing a nonlinear gain g such that $\epsilon b \equiv |g|e^{i\theta_b}$, where θ_b is the relative phase between the pump and the fundamental field.

Assuming $\Delta_{\text{cav}}^a = 0$ and that only the input port drives the cavity, Eq. (4.3.5) reduces to

$$\dot{\hat{a}} = -\kappa_a \hat{a} + |g|e^{i\theta_b} \hat{a}^\dagger + \sqrt{2\kappa_{\text{in}}^a} \hat{A}_{\text{in}}, \quad (4.3.8)$$

$$\dot{\hat{a}}^\dagger = -\kappa_a \hat{a}^\dagger + |g|e^{-i\theta_b} \hat{a} + \sqrt{2\kappa_{\text{in}}^a} \hat{A}_{\text{in}}^\dagger. \quad (4.3.9)$$

Considering the steady state ($\dot{\hat{a}} = 0$), the classical nonlinear gain in optical power is defined as

$$\mathcal{G} = \frac{P_{\text{out}}(g)}{P_{\text{out}}(g=0)} = \frac{1 + (|g|/\kappa_a)^2 + 2(|g|/\kappa_a) \cos \theta_b}{(1 - (|g|/\kappa_a)^2)^2}. \quad (4.3.10)$$

The gain is maximized for $\theta_b = 0$ (amplification) and minimized for $\theta_b = \pi$ (deamplification), and diverges as $|g| \rightarrow \kappa_a$. To simplify the following discussion, we introduce a normalized nonlinear gain factor x , defined by scaling the pump strength with the OPO threshold power P_{th} :

$$x \equiv \frac{|g|}{\kappa_a} = \sqrt{\frac{P}{P_{\text{th}}}}. \quad (4.3.11)$$

This dimensionless parameter x can be directly related to the squeezing parameter r as $x = \tanh r$. Using this parameter, Eq. (4.3.10) reduces, for $\theta_b = 0$ and π , to

$$\mathcal{G}_{\text{amp}} = \frac{1}{(1 - x)^2}, \quad (4.3.12)$$

$$\mathcal{G}_{\text{deamp}} = \frac{1}{(1 + x)^2}. \quad (4.3.13)$$

4.3.2 Semi-classical noise response in an OPO

We linearize the operators by decomposing them into their mean values and fluctuations, as in Eq. (4.1.2). The Langevin equations for the fluctuations, derived from Eq. (4.3.9), are then written as

$$\delta \dot{\hat{a}} = -(\kappa_a - i\Delta_{\text{cav}}^a) \delta \hat{a} + |g|e^{i\theta_b} \delta \hat{a}^\dagger + \sqrt{2\kappa_{\text{in}}^a} \delta \hat{A}_{\text{in}} + \sqrt{2\kappa_{\text{out}}^a} \delta \hat{A}_{\text{out}} + \sqrt{2\kappa_l^a} \delta \hat{A}_l, \quad (4.3.14)$$

$$\delta \dot{\hat{a}}^\dagger = -(\kappa_a + i\Delta_{\text{cav}}^a) \delta \hat{a}^\dagger + |g|e^{-i\theta_b} \delta \hat{a} + \sqrt{2\kappa_{\text{in}}^a} \delta \hat{A}_{\text{in}}^\dagger + \sqrt{2\kappa_{\text{out}}^a} \delta \hat{A}_{\text{out}}^\dagger + \sqrt{2\kappa_l^a} \delta \hat{A}_l^\dagger. \quad (4.3.15)$$

Applying a Fourier transform, these equations can be expressed as

$$i\omega \delta \tilde{\mathbf{a}} = \mathbf{M}_{\text{OPO}} \delta \tilde{\mathbf{a}} + \sqrt{2\kappa_{\text{in}}^a} \delta \tilde{\mathbf{A}}_{\text{in}} + \sqrt{2\kappa_{\text{out}}^a} \delta \tilde{\mathbf{A}}_{\text{out}} + \sqrt{2\kappa_l^a} \delta \tilde{\mathbf{A}}_l. \quad (4.3.16)$$

The matrix and vector components are defined as

$$\mathbf{M}_{\text{OPO}} = \begin{bmatrix} -\kappa_a + i\Delta_{\text{cav}}^a & \kappa_a x e^{i\theta_b} \\ \kappa_a x e^{-i\theta_b} & -\kappa_a - i\Delta_{\text{cav}}^a \end{bmatrix}, \quad (4.3.17)$$

$$\delta \tilde{\mathbf{a}} = \begin{bmatrix} \delta \tilde{a} \\ \delta \tilde{a}^\dagger \end{bmatrix}, \quad \delta \tilde{\mathbf{A}}_j = \begin{bmatrix} \delta \tilde{A}_j \\ \delta \tilde{A}_j^\dagger \end{bmatrix}. \quad (4.3.18)$$

The matrix $\mathbf{M}_{\text{OPO}}$ consists of diagonal terms describing the cavity dynamics and off-diagonal terms arising from the nonlinear interaction. Setting the relative phase to $\theta_b = 0$ and solving the system using the boundary conditions in Eq. (4.1.14), the quadrature response of the OPO can be written as

$$\begin{aligned}\delta\tilde{\mathbf{X}}_{\text{OPO}}^{\text{refl}} &= [\mathbf{\Gamma} (2\kappa_{\text{in}}^a(i\omega\mathbf{I} - \mathbf{M}_{\text{OPO}})^{-1} - \mathbf{I}) \mathbf{\Gamma}^{-1}] \delta\tilde{\mathbf{X}}_{\text{in}} \\ &\quad + 2\sqrt{\kappa_{\text{in}}^a\kappa_{\text{out}}^a} \mathbf{\Gamma} (i\omega\mathbf{I} - \mathbf{M}_{\text{OPO}})^{-1} \mathbf{\Gamma}^{-1} \delta\tilde{\mathbf{X}}_{\text{out}} \\ &\quad + 2\sqrt{\kappa_{\text{in}}^a\kappa_l^a} \mathbf{\Gamma} (i\omega\mathbf{I} - \mathbf{M}_{\text{OPO}})^{-1} \mathbf{\Gamma}^{-1} \delta\tilde{\mathbf{X}}_{l,\text{in}} \\ &= \mathbf{T}_{\text{OPO}}^{\text{refl}} \delta\tilde{\mathbf{X}}_{\text{in}} + \mathbf{T}_{\text{OPO}}^{\text{out}} \delta\tilde{\mathbf{X}}_{\text{out}} + \mathbf{T}_{\text{OPO}}^l \delta\tilde{\mathbf{X}}_{l,\text{in}}.\end{aligned}\tag{4.3.19}$$

The transfer matrices are given by

$$\begin{aligned}\mathbf{T}_{\text{OPO}}^{\text{refl}} &= \begin{bmatrix} \frac{2\kappa_{\text{in}}^a - \kappa_a + |g| - i(\omega - \Delta_{\text{cav}}^a)}{\kappa_a - |g| + i(\omega - \Delta_{\text{cav}}^a)} & 0 \\ 0 & \frac{2\kappa_{\text{in}}^a - \kappa_a - |g| - i(\omega - \Delta_{\text{cav}}^a)}{\kappa_a + |g| + i(\omega - \Delta_{\text{cav}}^a)} \end{bmatrix}, \\ \mathbf{T}_{\text{OPO}}^{\text{out}} &= \begin{bmatrix} \frac{2\sqrt{\kappa_{\text{in}}^a\kappa_{\text{out}}^a}}{\kappa_a - |g| + i(\omega - \Delta_{\text{cav}}^a)} & 0 \\ 0 & \frac{2\sqrt{\kappa_{\text{in}}^a\kappa_{\text{out}}^a}}{\kappa_a + |g| + i(\omega - \Delta_{\text{cav}}^a)} \end{bmatrix}, \\ \mathbf{T}_{\text{OPO}}^l &= \begin{bmatrix} \frac{2\sqrt{\kappa_{\text{in}}^a\kappa_l^a}}{\kappa_a - |g| + i(\omega - \Delta_{\text{cav}}^a)} & 0 \\ 0 & \frac{2\sqrt{\kappa_{\text{in}}^a\kappa_l^a}}{\kappa_a + |g| + i(\omega - \Delta_{\text{cav}}^a)} \end{bmatrix}.\end{aligned}\tag{4.3.20}$$

Furthermore, under resonant conditions ($\Delta_{\text{cav}}^a = 0$), the quadrature responses of the OPO are given by

$$\delta\tilde{\mathbf{X}}_{\text{OPO}}^{\text{refl}} = \frac{(2\kappa_{\text{in}}^a - \kappa_a + |g| - i\omega)\delta\tilde{\mathbf{X}}_{\text{in}} + 2\sqrt{\kappa_{\text{in}}^a\kappa_{\text{out}}^a}\delta\tilde{\mathbf{X}}_{\text{out}} + 2\sqrt{\kappa_{\text{in}}^a\kappa_l^a}\delta\tilde{\mathbf{X}}_{l,\text{in}}}{\kappa_a \mp |g| + i\omega},\tag{4.3.21}$$

$$\delta\tilde{\mathbf{X}}_{\text{OPO}}^{\text{trans}} = \frac{2\sqrt{\kappa_{\text{in}}^a\kappa_{\text{out}}^a}\delta\tilde{\mathbf{X}}_{\text{in}} + (2\kappa_{\text{in}}^a - \kappa_a - |g| - i\omega)\delta\tilde{\mathbf{X}}_{\text{out}} + 2\sqrt{\kappa_{\text{out}}^a\kappa_l^a}\delta\tilde{\mathbf{X}}_{l,\text{in}}}{\kappa_a \mp |g| + i\omega}.\tag{4.3.22}$$

We next evaluate the variance of the reflected quadrature fluctuations $\delta\tilde{\mathbf{X}}_{\text{OPO}}^{\text{refl}}$ from the input port. For simplicity, all ports other than the input port of interest are treated collectively as loss channels. Assuming that both the input fluctuations and the loss-induced fluctuations are in coherent vacuum states, $V(\delta\tilde{\mathbf{X}}_{\text{in}}) = V(\delta\tilde{\mathbf{X}}_{l,\text{in}}) = V_{\text{vac}}$, the variance is given by

$$V(\delta\tilde{\mathbf{X}}_{\text{OPO}}^{\text{refl}}) = \mathbf{T}_{\text{OPO}}^{\text{refl}} \mathbf{T}_{\text{OPO}}^{\text{refl}\dagger} V_{\text{vac}} + \mathbf{T}_{\text{OPO}}^l \mathbf{T}_{\text{OPO}}^{l\dagger} V_{\text{vac}}.\tag{4.3.23}$$

The resulting covariance matrix is therefore written as

$$\mathbf{V}_{\text{OPO}} = \begin{bmatrix} V_{\text{anti-sqz}} & 0 \\ 0 & V_{\text{sqz}} \end{bmatrix}.\tag{4.3.24}$$

The diagonal elements are explicitly given by

$$V_{\text{anti-sqz}} = 1 + \frac{\kappa_{\text{in}}^a}{\kappa_a} \frac{4|g|/\kappa_a}{(\omega/\kappa_a)^2 + (1 - |g|/\kappa_a)^2}, \quad (4.3.25)$$

$$V_{\text{sqz}} = 1 - \frac{\kappa_{\text{in}}^a}{\kappa_a} \frac{4|g|/\kappa_a}{(\omega/\kappa_a)^2 + (1 + |g|/\kappa_a)^2}. \quad (4.3.26)$$

For $\theta_b = 0$, the variance of the amplitude quadrature increases, while that of the phase quadrature decreases. In other words, the amplitude quadrature is anti-squeezed and the phase quadrature is squeezed. This demonstrates that the OPO generates correlations between the two quadratures, redistributing their uncertainties while preserving the Heisenberg uncertainty relation. In the absence of internal loss, i.e. $\kappa_l^a = 0$, the product of the quadrature variances satisfies

$$V_{\text{anti-sqz}} V_{\text{sqz}} = 1. \quad (4.3.27)$$

In this ideal case, as shown by the solid green curve in Fig. 4.7(a), the squeezing level rapidly improves with increasing nonlinear gain factor x . In contrast, when loss is present, the squeezing level saturates at a finite value even if the nonlinear gain is increased, and no further improvement can be achieved.

The escape efficiency quantifies how efficiently the squeezed field is extracted from the OPO cavity and characterizes internal losses in the OPO. It is defined in terms of the cavity decay rates as

$$\eta_{\text{esc}} = \frac{\kappa_{\text{in}}^a}{\kappa^a}. \quad (4.3.28)$$

Using the escape efficiency, Eq. (4.3.26) can be rewritten as

$$V_{\text{anti-sqz}} = 1 + \frac{4\eta_{\text{esc}}x}{(\omega/\kappa_a)^2 + (1 - x)^2}, \quad (4.3.29)$$

$$V_{\text{sqz}} = 1 - \frac{4\eta_{\text{esc}}x}{(\omega/\kappa_a)^2 + (1 + x)^2}. \quad (4.3.30)$$

Figure 4.7(b) shows the squeezing and anti-squeezing variances. The solid curve represents the squeezing variance, while the dashed curve represents the anti-squeezing variance. It is evident that the squeezing variance is more susceptible to optical loss than the anti-squeezing variance.

4.4 Nondegenerate OPO equation of motion

In this section, we analyze a nondegenerate optical parametric oscillator (NOPO), in which the signal and idler fields oscillate at different frequencies. An NOPO is known to generate quantum-correlated two-mode squeezed states [33]. We consider a down-conversion process in which a pump field at frequency $2\omega_0$ interacts with a nonlinear optical crystal. It produces an idler field at frequency $\omega_0 + \Lambda$ and a signal field at frequency $\omega_0 - \Lambda$.

The total Hamiltonian of the NOPO system, $H_{\text{tot}}^{\text{NOPO}}$, is given by

$$H_{\text{tot}}^{\text{NOPO}} = H_{\text{cav}}^{\text{NOPO}} + H_{\text{bath}}^{\text{NOPO}} + H_{\text{ext}}^{\text{NOPO}}, \quad (4.4.1)$$

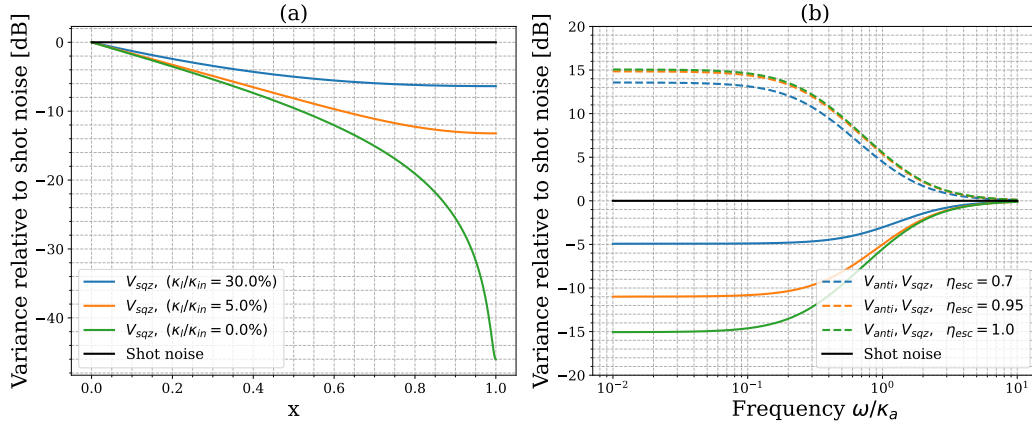


Figure 4.7: Quadrature variances $V_{\text{anti-sqz}}$ and V_{sqz} as a function of normalised frequency ω/κ_a

which consists of three contributions: the intracavity Hamiltonian, the bath Hamiltonian, and the interaction Hamiltonian describing the coupling between the cavity modes and the external fields. The intracavity Hamiltonian includes a three-wave mixing term responsible for the parametric interaction among the pump, signal, and idler modes.

The total Hamiltonian is composed of the following three terms:

- $H_{\text{cav}}^{\text{NOPO}}$: the Hamiltonian describing the closed system formed by the intracavity fields,

$$H_{\text{cav}}^{\text{NOPO}} = \hbar(\omega_0 + \Lambda)\hat{a}_+^\dagger\hat{a}_+ + \hbar(\omega_0 - \Lambda)\hat{a}_-^\dagger\hat{a}_- + 2\hbar\omega_0\hat{b}^\dagger\hat{b} + \frac{i\hbar}{2} \left(\epsilon\hat{a}_+^\dagger\hat{a}_-^\dagger\hat{b} - \epsilon^*\hat{a}_+\hat{a}_-\hat{b}^\dagger \right), \quad (4.4.2)$$

- $H_{\text{bath}}^{\text{NOPO}}$: the Hamiltonian describing the external bath fields coupled to the cavity,

$$H_{\text{bath}}^{\text{NOPO}} = \sum_j \int_{-\infty}^{+\infty} \hbar\omega \left(\hat{A}_j^{+\dagger}(\omega)\hat{A}_j^+(\omega) + \hat{A}_j^{-\dagger}(\omega)\hat{A}_j^-(\omega) + \hat{B}_j^\dagger(\omega)\hat{B}_j(\omega) \right) d\omega, \quad (4.4.3)$$

- $H_{\text{ext}}^{\text{NOPO}}$: the Hamiltonian describing the interaction between the cavity modes and the external bath fields,

$$H_{\text{ext}}^{\text{NOPO}} = i\hbar \sum_{j,\pm} \int_{-\infty}^{+\infty} \sqrt{2\kappa_j^\pm} \left(\hat{A}_j^{\pm\dagger}(\omega)\hat{a}_\pm - \hat{a}_\pm^\dagger\hat{A}_j^\pm(\omega) \right) d\omega. \quad (4.4.4)$$

Here, $\hat{a}_+$, $\hat{a}_-$, and $\hat{b}$ denote the intracavity modes corresponding to the idler, signal, and pump fields, respectively. The operators $\hat{A}_j^+$, $\hat{A}_j^-$, and $\hat{B}_j$ represent the corresponding external flux fields. The parameters $\kappa_j^\pm$ denote the decay rates associated with each coupling port, and ϵ is the nonlinear coupling coefficient.

From the total Hamiltonian given in Eq. (4.4.1), the Heisenberg–Langevin equations for the intracavity modes $\hat{a}_+$, $\hat{a}_-$, and $\hat{b}$ are obtained as

$$\begin{aligned}\dot{\hat{a}}_+ &= -(\kappa_{\text{cav}}^+ - i\Delta_{\text{cav}}^+)\hat{a}_+ + \frac{\epsilon}{2}\hat{b}\hat{a}_+^\dagger + \sqrt{2\kappa_{\text{in}}^+}\hat{A}_{\text{in}}^+ + \sqrt{2\kappa_{\text{out}}^+}\hat{A}_{\text{out}}^+ + \sqrt{2\kappa_l^+}\hat{A}_l^+, \\ \dot{\hat{a}}_- &= -(\kappa_{\text{cav}}^- - i\Delta_{\text{cav}}^-)\hat{a}_- + \frac{\epsilon}{2}\hat{b}\hat{a}_-^\dagger + \sqrt{2\kappa_{\text{in}}^-}\hat{A}_{\text{in}}^- + \sqrt{2\kappa_{\text{out}}^-}\hat{A}_{\text{out}}^- + \sqrt{2\kappa_l^-}\hat{A}_l^-, \\ \dot{\hat{b}} &= -(\kappa_b - i\Delta_{\text{cav}}^b)\hat{b} - \frac{\epsilon^*}{2}\hat{a}_+\hat{a}_- + \sqrt{2\kappa_{\text{in}}^b}\hat{B}_{\text{in}} + \sqrt{2\kappa_{\text{out}}^b}\hat{B}_{\text{out}} + \sqrt{2\kappa_l^b}\hat{B}_l.\end{aligned}\quad (4.4.5)$$

Here, the cavity detunings are defined as $\Delta_{\text{cav}}^\pm = \omega_0 - \omega \pm \Lambda$. In the following analysis, we impose the following two conditions:

- The cavity detunings are set to zero: $\Delta_{\text{cav}}^\pm \simeq \Delta_{\text{cav}}^a = 0$.
- The decay rates for the signal and idler modes are assumed to be identical: $\kappa_j^+ = \kappa_j^- = \kappa_j^a$.

Applying a Fourier transform to Eq. (4.4.5) and rewriting the equations in matrix form, we obtain

$$i\omega\tilde{\mathbf{a}} = \mathbf{M}_{\text{NOPO}}\tilde{\mathbf{a}} + \sqrt{2\kappa_{\text{in}}^a}\tilde{\mathbf{A}}_{\text{in}} + \sqrt{2\kappa_{\text{out}}^a}\tilde{\mathbf{A}}_{\text{out}} + \sqrt{2\kappa_l^a}\tilde{\mathbf{A}}_{l,\text{in}}. \quad (4.4.6)$$

The vectors appearing in the above equation are defined as

$$\tilde{\mathbf{a}} = \begin{bmatrix} \tilde{a}_+ \\ \tilde{a}_+^\dagger \\ \tilde{a}_- \\ \tilde{a}_-^\dagger \end{bmatrix}, \quad \tilde{\mathbf{A}}_j = \begin{bmatrix} \tilde{A}_j^+ \\ \tilde{A}_j^{+\dagger} \\ \tilde{A}_j^- \\ \tilde{A}_j^{-\dagger} \end{bmatrix}. \quad (4.4.7)$$

The matrix $\mathbf{M}_{\text{NOPO}}$ is given by

$$\mathbf{M}_{\text{NOPO}} = \kappa_a \begin{bmatrix} -1 & 0 & 0 & xe^{i\theta_b} \\ 0 & -1 & xe^{-i\theta_b} & 0 \\ 0 & xe^{i\theta_b} & -1 & 0 \\ xe^{-i\theta_b} & 0 & 0 & -1 \end{bmatrix}. \quad (4.4.8)$$

For simplicity, the output port and loss port are treated as a single effective loss channel. We linearize the system by decomposing each operator into its steady-state mean value and a fluctuation around it,

$$\tilde{\mathbf{a}} = \bar{\mathbf{a}} + \delta\tilde{\mathbf{a}}, \quad \tilde{\mathbf{A}}_j = \bar{\mathbf{A}}_j + \delta\tilde{\mathbf{A}}_j. \quad (4.4.9)$$

Using the boundary condition in Eq. (4.1.14), the linear response of the NOPO to fluctuation operators is obtained as

$$\begin{aligned}\delta\tilde{\mathbf{A}}_{\text{NOPO}}^{\text{refl}} &= [2\kappa_{\text{in}}^a(i\omega\mathbf{I} - \mathbf{M}_{\text{NOPO}})^{-1} - \mathbf{I}] \delta\tilde{\mathbf{A}}_{\text{in}} \\ &\quad + 2\sqrt{\kappa_{\text{in}}^a\kappa_l^a}(i\omega\mathbf{I} - \mathbf{M}_{\text{NOPO}})^{-1} \delta\tilde{\mathbf{A}}_{l,\text{in}}.\end{aligned}\quad (4.4.10)$$

Here, we introduce the transformation matrix $\mathbf{\Gamma}_2$ that converts the flux-field vector into the quadrature vector $\delta\tilde{\mathbf{X}}^j$,

$$\delta\tilde{\mathbf{X}}^j = \mathbf{\Gamma}_2 \delta\tilde{\mathbf{A}}_j, \quad (4.4.11)$$

$$\delta\tilde{\mathbf{X}}^j = \begin{bmatrix} \delta\tilde{X}_j^+ \\ \delta\tilde{X}_j^{+\dagger} \\ \delta\tilde{X}_j^- \\ \delta\tilde{X}_j^{-\dagger} \end{bmatrix}, \quad \mathbf{\Gamma}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -i & i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{bmatrix}. \quad (4.4.12)$$

Using this transformation, the quadrature response of the NOPO can be written as

$$\begin{aligned} \delta\tilde{\mathbf{X}}_{\text{NOPO}}^{\text{refl}} &= \mathbf{\Gamma}_2 [2\kappa_{\text{in}}^a (i\omega\mathbf{I} - \mathbf{M}_{\text{NOPO}})^{-1} - \mathbf{I}] \mathbf{\Gamma}_2^{-1} \delta\tilde{\mathbf{X}}_{\text{NOPO}}^{\text{in}} \\ &\quad + 2\sqrt{\kappa_{\text{in}}^a \kappa_l^a} \mathbf{\Gamma}_2 (i\omega\mathbf{I} - \mathbf{M}_{\text{NOPO}})^{-1} \mathbf{\Gamma}_2^{-1} \delta\tilde{\mathbf{X}}_{\text{NOPO}}^{l,\text{in}} \\ &= T_{\text{NOPO}}^{\text{refl}} \delta\tilde{\mathbf{X}}_{\text{NOPO}}^{\text{in}} + T_{\text{NOPO}}^l \delta\tilde{\mathbf{X}}_{\text{NOPO}}^{l,\text{in}}. \end{aligned} \quad (4.4.13)$$

Defining $\kappa_a = \kappa_{\text{in}}^a + \kappa_l^a$ and the escape efficiency $\eta_{\text{esc}} = \kappa_{\text{in}}^a / \kappa_a$, the matrix elements are given by

$$T_{\text{NOPO}}^{\text{refl}} = \begin{bmatrix} T_d & 0 & T_{\text{od}} \cos \theta_b & T_{\text{od}} \sin \theta_b \\ 0 & T_d & T_{\text{od}} \sin \theta_b & -T_{\text{od}} \cos \theta_b \\ T_{\text{od}} \cos \theta_b & T_{\text{od}} \sin \theta_b & T_d & 0 \\ T_{\text{od}} \sin \theta_b & -T_{\text{od}} \cos \theta_b & 0 & T_d \end{bmatrix}, \quad (4.4.14)$$

$$T_{\text{NOPO}}^l = \begin{bmatrix} T_d^l & 0 & T_{\text{od}}^l \cos \theta_b & T_{\text{od}}^l \sin \theta_b \\ 0 & T_d^l & T_{\text{od}}^l \sin \theta_b & -T_{\text{od}}^l \cos \theta_b \\ T_{\text{od}}^l \cos \theta_b & T_{\text{od}}^l \sin \theta_b & T_d^l & 0 \\ T_{\text{od}}^l \sin \theta_b & -T_{\text{od}}^l \cos \theta_b & 0 & T_d^l \end{bmatrix}. \quad (4.4.15)$$

The scalar coefficients appearing in these matrices are given by

$$T_d = -1 + \frac{2\eta_{\text{esc}}(1 + i\omega/\kappa_a)}{(1 + i\omega/\kappa_a)^2 - x^2}, \quad T_{\text{od}} = \frac{2\eta_{\text{esc}}x}{(1 + i\omega/\kappa_a)^2 - x^2}, \quad (4.4.16)$$

$$T_d^l = \frac{2\sqrt{\eta_{\text{esc}}(1 - \eta_{\text{esc}})}(1 + i\omega/\kappa_a)}{(1 + i\omega/\kappa_a)^2 - x^2}, \quad T_{\text{od}}^l = \frac{2\sqrt{\eta_{\text{esc}}(1 - \eta_{\text{esc}})}x}{(1 + i\omega/\kappa_a)^2 - x^2}. \quad (4.4.17)$$

We assume that coherent vacuum states are injected into both the input and loss ports of the OPO, such that

$$V(\delta\tilde{\mathbf{X}}_{\text{NOPO}}^{\text{in}}) = V(\delta\tilde{\mathbf{X}}_{\text{NOPO}}^{l,\text{in}}) = V_{\text{vac}} = 1. \quad (4.4.18)$$

Following the same procedure as in Eq. (4.3.23), the covariance matrix is calculated as

$$V_{\text{NOPO}} = \begin{bmatrix} V(\delta\tilde{X}_1^+) & V(\delta\tilde{X}_1^+\delta\tilde{X}_2^+) & V(\delta\tilde{X}_1^+\delta\tilde{X}_1^-) & V(\delta\tilde{X}_1^+\delta\tilde{X}_2^-) \\ V(\delta\tilde{X}_2^+\delta\tilde{X}_1^+) & V(\delta\tilde{X}_2^+) & V(\delta\tilde{X}_2^+\delta\tilde{X}_1^-) & V(\delta\tilde{X}_2^+\delta\tilde{X}_2^-) \\ V(\delta\tilde{X}_1^-\delta\tilde{X}_1^+) & V(\delta\tilde{X}_1^-\delta\tilde{X}_2^+) & V(\delta\tilde{X}_1^-) & V(\delta\tilde{X}_1^-\delta\tilde{X}_2^-) \\ V(\delta\tilde{X}_2^-\delta\tilde{X}_1^+) & V(\delta\tilde{X}_2^-\delta\tilde{X}_2^+) & V(\delta\tilde{X}_2^-\delta\tilde{X}_1^-) & V(\delta\tilde{X}_2^-) \end{bmatrix} \quad (4.4.19)$$

$$= \begin{bmatrix} V_{11} & 0 & V_{13} \cos \theta_b & V_{14} \sin \theta_b \\ 0 & V_{22} & V_{23} \sin \theta_b & -V_{24} \cos \theta_b \\ V_{31} \cos \theta_b & V_{32} \sin \theta_b & V_{33} & 0 \\ V_{41} \sin \theta_b & -V_{42} \cos \theta_b & 0 & V_{44} \end{bmatrix}. \quad (4.4.20)$$

Since the diagonal and off-diagonal elements are equal, we define

$$V_{11} = V_{22} = V_{33} = V_{44} \equiv V_d, \quad (4.4.21)$$

$$V_{13} = V_{14} = V_{23} = V_{24} = V_{31} = V_{41} = V_{32} = V_{42} \equiv V_{od}. \quad (4.4.22)$$

The matrix elements are then expressed as

$$V_d = 1 + \frac{8\eta_{\text{esc}}x^2}{(1 - 2x + x^2 + \omega^2/\kappa^2)(1 + 2x + x^2 + \omega^2/\kappa^2)}, \quad (4.4.23)$$

$$V_{od} = \frac{4\eta_{\text{esc}}x(1 + x^2 + \omega^2/\kappa^2)}{(1 - 2x + x^2 + \omega^2/\kappa^2)(1 + 2x + x^2 + \omega^2/\kappa^2)}. \quad (4.4.24)$$

Figure 4.8(a) shows the variances of the diagonal and off-diagonal elements of the covariance matrix. Both variances exceed the shot-noise level within the cavity linewidth, indicating that no squeezing appears in either the signal or idler mode individually.

To reveal quantum correlations between the two modes, we consider projection measurements of the quadratures. The signal and idler quadratures are projected at angles ζ_s and ζ_i , respectively, and the variances of the combined measurement outcomes are evaluated. As discussed in detail in Sec. 4.5.2, ζ_s and ζ_i correspond to the homodyne angles for the signal and idler fields.

We define the projection vector onto the measured quadratures as

$$\mathbf{H}_{\zeta_i, \zeta_s} = [\cos \zeta_i, \sin \zeta_i, \cos \zeta_s, \sin \zeta_s]. \quad (4.4.25)$$

The combined quadrature variance is then given by

$$V_{\zeta_i, \zeta_s} = \mathbf{H}_{\zeta_i, \zeta_s} V_{\text{NOPO}} \mathbf{H}_{\zeta_i, \zeta_s}^T \quad (4.4.26)$$

$$= 2V_d + 2V_{od} \cos \theta_b (\cos \zeta_s \cos \zeta_i - \sin \zeta_s \sin \zeta_i) \\ + 2V_{od} \sin \theta_b (\cos \zeta_s \sin \zeta_i + \sin \zeta_s \cos \zeta_i). \quad (4.4.27)$$

For $\zeta_i = \zeta_s = 0$, the variance of the sum of the amplitude quadratures, $a_1^i + a_1^s$, is measured and is denoted by $V_{i_1+s_1}$. Similarly, for $\zeta_i = 0$ and $\zeta_s = \pi$, the variance of the difference

$a_1^i - a_1^s$ is obtained and denoted by $V_{i_1-s_1}$:

$$V_{i_1+s_1} = V_{i_2-s_2} = 2(V_d + V_{od} \cos \theta_b), \quad (4.4.28)$$

$$V_{i_1-s_1} = V_{i_2+s_2} = 2(V_d - V_{od} \cos \theta_b). \quad (4.4.29)$$

The variances of the sum and difference of the phase quadratures, $a_2^i \pm a_2^s$, can also be measured by choosing appropriate values of ζ_i and ζ_s . For $\theta_b = 0$, the combined variances reduce to

$$V_{i_1+s_1} = V_{i_2-s_2} = 2 \left(1 + \frac{4\eta_{\text{esc}} x}{(\omega/\kappa_a)^2 + (1-x)^2} \right) = 2V_{\text{anti-sqz}}, \quad (4.4.30)$$

$$V_{i_1-s_1} = V_{i_2+s_2} = 2 \left(1 - \frac{4\eta_{\text{esc}} x}{(\omega/\kappa_a)^2 + (1+x)^2} \right) = 2V_{\text{sqz}}. \quad (4.4.31)$$

These results show that, although neither the signal nor the idler mode is individually squeezed, the joint quadratures exhibit squeezing due to their quantum correlations.

Compared with the degenerate OPO case given in Eq. (4.3.30), the variances in the nondegenerate case are larger by an overall factor of two. This is because the signal and idler fields correspond to two independent modes, each contributing vacuum fluctuations at the shot-noise level. When the measurement outcomes of the two modes are combined, the vacuum noise from both modes is added, resulting in an overall degradation of the squeezing level by $10 \log_{10} 2 \simeq 3 \text{ dB}$ compared with that of a single-mode degenerate OPO. Figure 4.8(b) compares the combined variance obtained from the NOPO with the squeezing variance of a degenerate OPO.

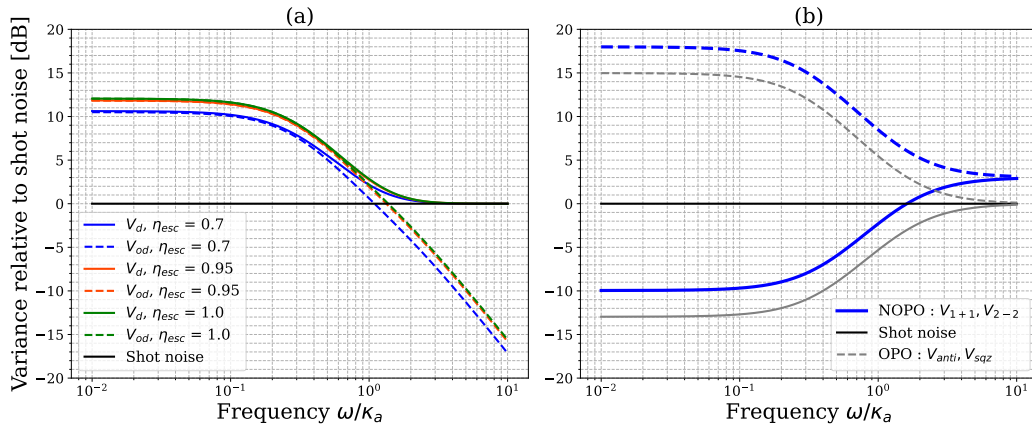


Figure 4.8: (a) Quadrature variances $V_{\text{anti-sqz}}$ and V_{sqz} as functions of the normalized frequency ω/κ_a . (b) Comparison between the squeezing variance obtained from a degenerate OPO and the combined variance obtained from a nondegenerate OPO. In both cases, the escape efficiency is set to $\eta_{\text{esc}} = 0.98$. For the nondegenerate OPO, the noise level is increased by 3 dB over the entire frequency range.

4.5 Measuring States of Light

In this section, we describe methods for detecting optical fields. We first discuss direct photodetection, in which the optical intensity is measured directly. We then introduce homodyne and heterodyne detection schemes, which enable the measurement of optical quadratures through interference with a local oscillator.

4.5.1 Direct photodetection

In general, an optical field is detected by converting light into an electrical signal via the photoelectric effect. The quantum efficiency η_{PD} is defined as the amount of electric charge generated per incident photon on a photodiode. When measuring quantum states of light, it is essential to use photodiodes with high quantum efficiency. In terms of the quantum efficiency η_{PD} , the photocurrent generated by a photodetector is given by

$$i = \frac{e\eta_{\text{PD}}}{\hbar\omega} P, \quad (4.5.1)$$

where e is the elementary charge, $\hbar\omega$ is the energy of a single photon, and $P = \hbar\omega N = \hbar\omega \langle \hat{a}^\dagger \hat{a} \rangle$ is the incident optical power.

To investigate the relationship between direct photodetection and quantum fluctuations, we linearize the field operators by decomposing the annihilation and creation operators into their steady-state mean values and fluctuation components:

$$\hat{a} = \bar{a} + \delta\hat{a}, \quad \hat{a}^\dagger = \bar{a}^* + \delta\hat{a}^\dagger.$$

Using this decomposition, the photocurrent operator can be written as⁵:

$$\begin{aligned} \hat{I} &= e\eta_{\text{PD}}(\hat{a}^\dagger \hat{a}) \\ &= e\eta_{\text{PD}} \left((\bar{a}^* + \delta\hat{a}^\dagger)(\bar{a} + \delta\hat{a}) \right) \\ &= e\eta_{\text{PD}} \left(|\bar{a}|^2 + |\bar{a}|(\delta\hat{a}e^{i\theta} + \delta\hat{a}^\dagger e^{-i\theta}) + \delta\hat{a}^\dagger \delta\hat{a} \right) \\ &= e\eta_{\text{PD}} \left(|\bar{a}|^2 + |\bar{a}| \delta\hat{X}_\theta + \delta\hat{a}^\dagger \delta\hat{a} \right), \end{aligned} \quad (4.5.2)$$

where the complex amplitude of the coherent steady-state field is written as $\bar{a} = |\bar{a}|e^{i\theta}$. The first term corresponds to the mean photocurrent, while the second term represents the contribution from the quadrature fluctuation $\delta\hat{X}_\theta$ defined by the phase θ of the coherent field. In practical photodetection systems, the photocurrent is converted into a voltage signal by a transimpedance amplifier (TIA). Denoting the transimpedance gain by R_T , the output voltage is given by $\hat{V} = R_T \hat{I}$. The variance of the voltage signal is therefore expressed as

$$\begin{aligned} V(\hat{V}) &= (eR_T\eta_{\text{PD}}|\bar{a}|^2)^2 \left(\langle \delta\hat{X}_\theta^2 \rangle - \langle \delta\hat{X}_\theta \rangle^2 \right) \\ &= (eR_T\eta_{\text{PD}}|\bar{a}|^2)^2 \langle \delta\hat{X}_\theta^2 \rangle, \end{aligned} \quad (4.5.3)$$

⁵Here we use the relation $\hat{X}_\theta = (\delta\hat{a}e^{i\theta} + \delta\hat{a}^\dagger e^{-i\theta}) = \hat{X}_1 \cos \theta + \hat{X}_2 \sin \theta$.

where second-order fluctuation terms have been neglected ($\delta\hat{a}^\dagger\delta\hat{a} = 0$).

This result shows that, in direct photodetection, the variance of the photocurrent depends only on the quadrature fluctuations that are in phase with the steady-state field. As a consequence, fluctuations in the orthogonal quadrature cannot be accessed by direct detection alone. This limitation motivates the use of homodyne detection, which enables access to arbitrary quadrature components of the optical field.

4.5.2 Homodyne detection

Homodyne detection is a measurement technique that enables access to arbitrary quadratures of an optical field and is widely used for characterizing squeezed states. As shown in Fig. 4.9, the signal field and a strong local oscillator are mixed at a beam splitter and detected by two photodetectors. By taking the difference between the photocurrents from the two detectors, classical noise contributions are canceled, allowing the quantum fluctuations appearing in the beat signal between the signal and the local oscillator to be measured. In this subsection, we derive the signal obtained in homodyne detection.

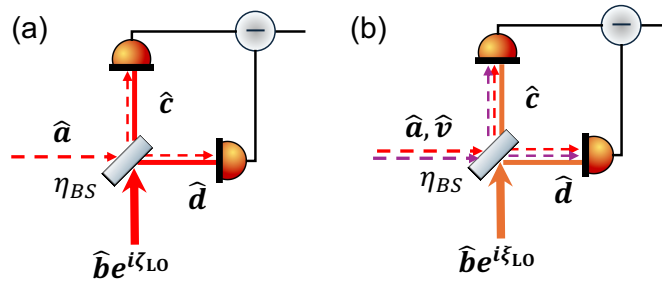


Figure 4.9: (a) Schematic of homodyne detection and (b) heterodyne detection

Let $\hat{a}$ denote the signal field and $\hat{b}$ the local oscillator field. Assuming a beam splitter with amplitude reflectivity $\sqrt{\eta_{BS}}$, the fields incident on the two photodetectors, $\hat{c}$ and $\hat{d}$, are given by

$$\begin{aligned}\hat{c} &= \sqrt{\eta_{BS}} \hat{a} + \sqrt{1 - \eta_{BS}} \hat{b} e^{i\zeta_{LO}}, \\ \hat{d} &= \sqrt{1 - \eta_{BS}} \hat{a} - \sqrt{\eta_{BS}} \hat{b} e^{i\zeta_{LO}},\end{aligned}\quad (4.5.4)$$

where ζ_{LO} is the relative phase between the signal and the local oscillator. Here, $\hat{c}$ and $\hat{d}$ represent annihilation operators of the optical fields incident on the detectors, rather than quadrature operators.

The photocurrents generated by the incident fields are given by $I_c = e\eta_{PD}\hat{c}^\dagger\hat{c}$ and $I_d = e\eta_{PD}\hat{d}^\dagger\hat{d}$, which can be written as

$$\begin{aligned}I_c &= e\eta_{PD} \left(\eta_{BS} \hat{a}^\dagger \hat{a} + (1 - \eta_{BS}) \hat{b}^\dagger \hat{b} + \sqrt{\eta_{BS}(1 - \eta_{BS})} (\hat{a}^\dagger \hat{b} e^{i\zeta_{LO}} + \hat{b}^\dagger \hat{a} e^{-i\zeta_{LO}}) \right), \\ I_d &= e\eta_{PD} \left((1 - \eta_{BS}) \hat{a}^\dagger \hat{a} + \eta_{BS} \hat{b}^\dagger \hat{b} - \sqrt{\eta_{BS}(1 - \eta_{BS})} (\hat{a}^\dagger \hat{b} e^{i\zeta_{LO}} + \hat{b}^\dagger \hat{a} e^{-i\zeta_{LO}}) \right).\end{aligned}\quad (4.5.5)$$

We decompose the field operators into their mean values and fluctuation components as $\hat{a} = \bar{a} + \delta\hat{a}$ and $\hat{b} = \bar{b} + \delta\hat{b}$. Assuming a balanced beam splitter ($\eta_{BS} = 0.5$), the differential

photocurrent is given by

$$I_c - I_d = e\eta_{PD} \left(2\bar{a}\bar{b} \cos \zeta_{LO} + \bar{b}(\delta\hat{a}^\dagger e^{i\zeta_{LO}} + \delta\hat{a} e^{-i\zeta_{LO}}) + \bar{a}(\delta\hat{b} e^{i\zeta_{LO}} + \delta\hat{b}^\dagger e^{-i\zeta_{LO}}) \right). \quad (4.5.6)$$

Furthermore, if the mean amplitude of the signal field is much smaller than that of the local oscillator ($\bar{b} \gg \bar{a}$), the differential photocurrent reduces to⁶:

$$I_{\text{Hom}}^- = I_c - I_d = e\eta_{PD} \bar{b} \left(\delta\hat{X}_1^a \cos \zeta_{LO} - \delta\hat{X}_2^a \sin \zeta_{LO} \right). \quad (4.5.7)$$

This expression shows that homodyne detection corresponds to a projection measurement of the signal quadrature onto the phase defined by the local oscillator. By controlling the relative phase ζ_{LO} between the signal and the local oscillator, an arbitrary quadrature of the signal field can be selectively measured.

The mean value of the differential photocurrent is zero, and its variance is given by

$$V(I_{\text{Hom}}^-) = |\bar{b}|^2 \left(\langle \delta\hat{X}_1^2 \rangle \cos^2 \zeta_{LO} + \langle \delta\hat{X}_2^2 \rangle \sin^2 \zeta_{LO} \right). \quad (4.5.8)$$

Finally, homodyne detection can be represented as a vector acting on the quadrature space,

$$\mathbf{H}_\zeta^{\text{Hom}} = [\cos \zeta, \sin \zeta], \quad (4.5.9)$$

which explicitly shows that the measured quadrature is determined by the local oscillator phase.

Relation between power spectral density and variance

The differential signal obtained from the homodyne detector is sent to a spectrum analyzer (SA), where it is measured as a power spectral density in the frequency domain. In general, the measurement bandwidth of the spectrum analyzer is set by the resolution bandwidth RBW . The variance expressed in Eq. (4.5.8) can be experimentally obtained by normalizing the measured power spectral density by the resolution bandwidth RBW and the local oscillator power $|\bar{b}|^2$:

$$V(\omega) = \frac{S_B(\omega)}{RBW \times |\bar{b}|^2}. \quad (4.5.10)$$

When the experimentally measured variance satisfies $V(\Omega) < 1$, the optical field is said to be in a squeezed state within the measurement bandwidth.

4.5.3 Heterodyne detection

Heterodyne detection is a measurement technique that exploits interference between a signal field and a local oscillator at a different optical frequency. Unlike homodyne

⁶Here we use the relation $\delta\hat{X}_\theta = \delta\hat{a}e^{i\theta} + \delta\hat{a}^\dagger e^{-i\theta} = \delta\hat{X}_1 \cos \theta + \delta\hat{X}_2 \sin \theta$.

detection, heterodyne detection allows simultaneous access to both quadrature components of the optical field. By introducing a frequency offset ω_H between the signal and the local oscillator and demodulating the detected signal at ω_H , information about the optical field can be extracted. A schematic of the heterodyne detection setup is shown in Fig. 4.9.

In the following, the frequency of the local oscillator field $\hat{b}e^{i\xi_{LO}}$ is taken to be ω_0 . We define the signal field at frequency $\omega_0 + \omega_H$ as $\hat{a}$ and the image-band vacuum field at frequency $\omega_0 - \omega_H$ as $\hat{v}$:

$$\begin{aligned}\hat{b} &= (\bar{b} + \delta\hat{b})e^{i(\omega_0 t + \xi_{LO})}, \\ \hat{a} &= (\bar{a} + \delta\hat{a})e^{i(\omega_0 + \omega_H)t}, \\ \hat{v} &= \delta\hat{v}e^{i(\omega_0 - \omega_H)t}.\end{aligned}\tag{4.5.11}$$

The optical fields incident on the two photodetectors, $\hat{c}$ and $\hat{d}$, are then given by

$$\begin{aligned}\hat{c} &= \frac{1}{\sqrt{2}} \left\{ (\bar{b} + \delta\hat{b})e^{i(\omega_0 t + \xi_{LO})} + (\bar{a} + \delta\hat{a})e^{i(\omega_0 + \omega_H)t} + \delta\hat{v}e^{i(\omega_0 - \omega_H)t} \right\}, \\ \hat{d} &= \frac{1}{\sqrt{2}} \left\{ -(\bar{b} + \delta\hat{b})e^{i(\omega_0 t + \xi_{LO})} + (\bar{a} + \delta\hat{a})e^{i(\omega_0 + \omega_H)t} + \delta\hat{v}e^{i(\omega_0 - \omega_H)t} \right\}.\end{aligned}\tag{4.5.12}$$

Here, ξ_{LO} denotes the relative phase between the signal and the local oscillator.

The photocurrents generated by the incident fields are given by $I_c = e\eta_{PD}\hat{c}^\dagger\hat{c}$ and $I_d = e\eta_{PD}\hat{d}^\dagger\hat{d}$, respectively, leading to

$$\begin{aligned}I_c &= \frac{1}{2}e\eta_{PD} \left(|\bar{b}|^2 + |\bar{a}|^2 + \hat{b}\delta\hat{X}_1^b + \hat{a}\delta\hat{X}_1^a + 2\bar{a}\bar{b}\cos(\omega_H + \xi_{LO}) \right. \\ &\quad + \bar{b}(\delta\hat{X}_1^a\cos(\omega_H - \xi_{LO}) + \delta\hat{X}_2^a\sin(\omega_H - \xi_{LO}) + \delta\hat{X}_1^v\cos(\omega_H + \xi_{LO}) - \delta\hat{X}_2^v\sin(\omega_H + \xi_{LO})) \\ &\quad \left. + \bar{a}(\delta\hat{X}_1^b\cos(\omega_H - \xi_{LO}) - \delta\hat{X}_2^b\sin(\omega_H - \xi_{LO}) + \delta\hat{X}_1^v\cos 2\omega_H - \delta\hat{X}_2^v\sin 2\omega_H) \right), \\ I_d &= \frac{1}{2}e\eta_{PD} \left(|\bar{b}|^2 + |\bar{a}|^2 + \hat{b}\delta\hat{X}_1^b + \hat{a}\delta\hat{X}_1^a - 2\bar{a}\bar{b}\cos(\omega_H + \xi_{LO}) \right. \\ &\quad - \bar{b}(\delta\hat{X}_1^a\cos(\omega_H - \xi_{LO}) + \delta\hat{X}_2^a\sin(\omega_H - \xi_{LO}) + \delta\hat{X}_1^v\cos(\omega_H + \xi_{LO}) - \delta\hat{X}_2^v\sin(\omega_H + \xi_{LO})) \\ &\quad \left. - \bar{a}(\delta\hat{X}_1^b\cos(\omega_H - \xi_{LO}) - \delta\hat{X}_2^b\sin(\omega_H - \xi_{LO}) - \delta\hat{X}_1^v\cos 2\omega_H + \delta\hat{X}_2^v\sin 2\omega_H) \right).\end{aligned}\tag{4.5.13}$$

The differential photocurrent is therefore written as

$$\begin{aligned}I_c - I_d &= e\eta_{PD} \left\{ 2\bar{a}\bar{b}\cos(\omega_H - \xi_{LO}) + \bar{a} \left(\delta\hat{X}_1^b\cos(\omega_H - \xi_{LO}) - \delta\hat{X}_2^b\sin(\omega_H - \xi_{LO}) \right) \right. \\ &\quad \left. + \bar{b} \left(\delta\hat{X}_1^a\cos(\omega_H - \xi_{LO}) + \delta\hat{X}_2^a\sin(\omega_H - \xi_{LO}) + \delta\hat{X}_1^v\cos(\omega_H + \xi_{LO}) - \delta\hat{X}_2^v\sin(\omega_H + \xi_{LO}) \right) \right\}.\end{aligned}\tag{4.5.14}$$

By demodulating the differential photocurrent $I_c - I_d$ with a cosine signal $\cos(\omega_H - \phi)$ and filtering out the high-frequency components, the heterodyne-demodulated signal is

obtained as

$$I_{\text{demod}}^{\text{Het}} = \frac{e\eta_{\text{PD}}}{2} \left[2\bar{a}\bar{b} \cos(\xi_{\text{LO}} - \phi) + \bar{a} \left(\delta\tilde{X}_1^b \cos(\xi_{\text{LO}} - \phi) + \tilde{X}_2^b \sin(\xi_{\text{LO}} - \phi) \right) \right. \\ \left. + \bar{b} \left(\delta\tilde{X}_1^a \cos(\xi_{\text{LO}} - \phi) - \delta\tilde{X}_2^a \sin(\xi_{\text{LO}} - \phi) + \delta\tilde{X}_1^v \cos(\xi_{\text{LO}} + \phi) - \delta\tilde{X}_2^v \sin(\xi_{\text{LO}} + \phi) \right) \right]. \quad (4.5.15)$$

Here, the annihilation operators $\hat{a}$ and $\hat{v}$ have been transformed into the corresponding quadrature operators according to

$$[\delta\hat{X}_1^a, \delta\hat{X}_2^a]^T = [\hat{a}^\dagger + \hat{a}, -i(\hat{a}^\dagger - \hat{a})]^T, \quad [\delta\hat{X}_1^v, \delta\hat{X}_2^v]^T = [\hat{v}^\dagger + \hat{v}, -i(\hat{v}^\dagger - \hat{v})]^T.$$

Finally, heterodyne detection can be represented as a vector acting on the quadrature space as

$$\mathbf{H}_{\xi_{\text{LO}}, \phi}^{\text{Het}} = \frac{1}{\sqrt{2}} [\cos(\xi_{\text{LO}} - \phi), -\sin(\xi_{\text{LO}} - \phi), \cos(\xi_{\text{LO}} + \phi), -\sin(\xi_{\text{LO}} + \phi)], \quad (4.5.16)$$

As an example, setting the local oscillator phase to $\xi_{\text{LO}} = \pi/2$, the detected voltages for in-phase ($\phi = -\pi/2$) and quadrature-phase ($\phi = \pi$) demodulation are given by

$$V_I^{\text{Het}} = \frac{eR_T\eta_{\text{PD}}}{2} \left[2\bar{a}\bar{b} + \bar{b}(-\delta\tilde{X}_1^a + \delta\tilde{X}_1^v) + \bar{a}\delta\tilde{X}_1^b \right], \\ V_Q^{\text{Het}} = \frac{eR_T\eta_{\text{PD}}}{2} \left[\bar{b}(\delta\tilde{X}_2^a + \delta\tilde{X}_2^v) + \bar{a}\delta\tilde{X}_2^b \right], \quad (4.5.17)$$

where R_T denotes the transimpedance gain of the detector. It is worth noting that heterodyne detection can be interpreted as a continuous-variable Bell-type measurement, in which the joint quadratures of the signal and vacuum modes are simultaneously accessed.

4.6 Limits to detecting squeezed states

4.6.1 Loss

Squeezed states are inevitably degraded by various loss mechanisms and are therefore detected in a reduced form. As the total loss increases, the observable squeezing level decreases and approaches that of a vacuum state. Typical sources of loss include optical propagation loss, mode mismatch, and imperfect detection efficiency. Such losses can be modeled as a partially transmitting mirror with an intensity transmission η . When loss is present in the optical field $\hat{a}$, vacuum fluctuations $\hat{v}$ enter the system. The output field and its variance are then given by

$$\hat{a}_{\text{out}} = \sqrt{\eta_l} \hat{a}_{\text{in}} + \sqrt{1 - \eta_l} \hat{v}, \\ V_{\text{out}} = \eta_l V_{\text{in}} + (1 - \eta_l) V_{\text{vac}}. \quad (4.6.1)$$

In particular, the squeezing and anti-squeezing variances given in Eq. (4.3.30) can be rewritten as

$$\begin{aligned} V_{\text{sqz}} &= 1 - \frac{4x\eta}{(\omega/\kappa_a)^2 + (1+x)^2}, \\ V_{\text{anti-sqz}} &= 1 + \frac{4x\eta}{(\omega/\kappa_a)^2 + (1-x)^2}. \end{aligned} \quad (4.6.2)$$

Here, the total detection efficiency η consists of several independent contributions:

- η_{esc} : escape efficiency of the OPO,
- η_{opt} : optical propagation efficiency,
- η_{vis} : mode matching (visibility) between the local oscillator and the squeezed field in homodyne detection,
- η_{QE} : quantum efficiency of the photodiodes.

4.6.2 Phase noise

Phase noise arises from relative fluctuations between the squeezing angle and the measurement angle. Such fluctuations couple the squeezed quadrature to the anti-squeezed quadrature, resulting in degradation of the observed squeezing level. The effect of phase noise on the squeezing variance can be expressed as

$$V_{\text{sqz}}^{\text{eff}} = V_{\text{sqz}} \cos^2 \bar{\theta} + V_{\text{anti-sqz}} \sin^2 \bar{\theta}, \quad (4.6.3)$$

where $\bar{\theta}$ denotes the root-mean-square value of the relative phase fluctuations. Figure 4.10 shows the squeezing level when phase noise is taken into account. While the anti-squeezing level is relatively insensitive to phase noise, the squeezing level is strongly affected due to coupling from the anti-squeezed quadrature. In particular, at high nonlinear gain, the squeezing level is significantly degraded, indicating that suppression of phase noise is essential for achieving high levels of squeezing.

Including both loss and phase noise, the effective squeezing variance is given by

$$V_{\text{sqz}}^{\text{eff}} = \eta (V_{\text{sqz}} \cos^2 \bar{\theta} + V_{\text{anti-sqz}} \sin^2 \bar{\theta}) + (1 - \eta) V_{\text{vac}}. \quad (4.6.4)$$

This expression explicitly shows how optical loss and phase noise jointly limit the achievable squeezing level in practical experiments.

4.7 Control systems

In this section, we describe a control technique used in the experiment. In general, operating optical cavities near resonance or maintaining an interferometer at the dark fringe is not straightforward due to environmental disturbances acting on the experimental setup. Feedback control is therefore employed using signals that are linear with respect to the parameters to be controlled. In this section, we introduce control methods commonly

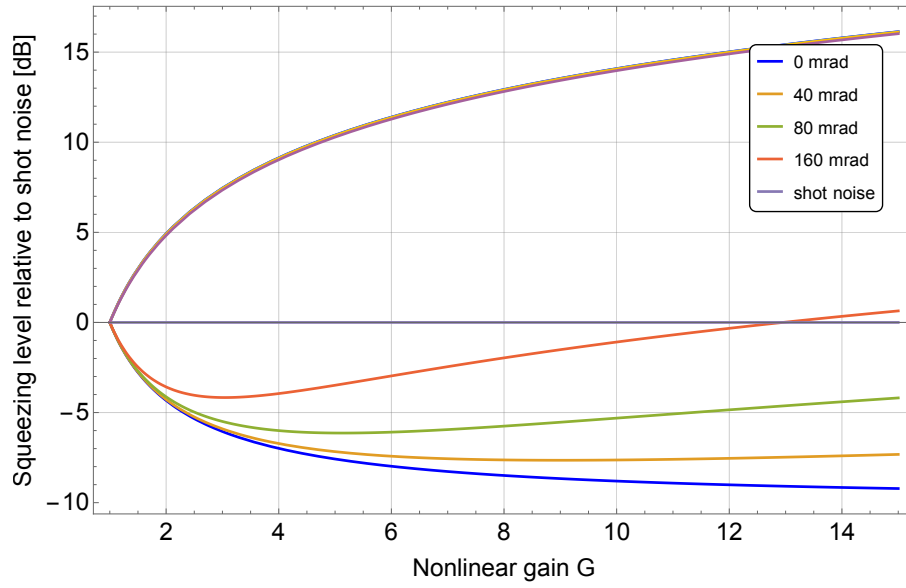


Figure 4.10: Squeezing level including the effect of phase noise. The total efficiency is set to $\eta = 0.98$, and the nonlinear gain factor is $g = 2$.

used to stabilize experimental parameters such as frequency, relative phase, and cavity length.

4.7.1 Pound–Drever–Hall method

We introduce the Pound–Drever–Hall (PDH) technique as a method for obtaining an error signal to control the cavity length [34]. The PDH method generates a linear error signal around the cavity resonance by applying phase modulation to the carrier field to produce radio-frequency (RF) sidebands and subsequently demodulating the reflected signal.

A schematic of the PDH setup is shown in Fig. 4.11.

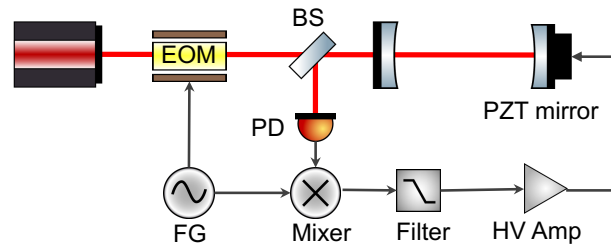


Figure 4.11: Schematic of the PDH method. A portion of the light reflected from the cavity is separated by a beam splitter and detected by a photodetector (PD). Phase modulation and demodulation are performed using a function generator. The detected signal and the local oscillator are mixed using a mixer, and high-frequency components are removed by a low-pass filter. The resulting error signal is amplified by a high-voltage amplifier and fed back to a piezoelectric transducer (PZT) to control the cavity length.

Phase modulation is performed using an electro-optic modulator (EOM). The refractive index of the EOM crystal changes in response to an applied voltage, inducing a phase

modulation by effectively varying the optical path length. When an electric field $E_0 e^{i\omega t}$ experiences an optical path length modulation of $X_0 \cos \omega_m t$ inside the EOM crystal, the transmitted electric field $E(t)$ is given by

$$E(t) = E_0 \exp\left(i\omega t + i\omega \frac{X_0 \cos \omega_m t}{c}\right) = E_0 \exp(i\omega t + im_m \cos \omega_m t), \quad (4.7.1)$$

where $m_m (\ll 1)$ is the modulation index determined by the applied voltage and the modulation efficiency of the crystal, and ω_m is the modulation frequency.

Using the generating function of the Bessel functions,

$$\sum_{n=-\infty}^{\infty} J_n(x) t^n = \exp\left[\frac{x}{2} \left(t - \frac{1}{t}\right)\right], \quad (4.7.2)$$

and substituting $x = m_m$ and $t = ie^{i\omega_m t}$, the electric field can be expanded as

$$E(t) = E_0 e^{i\omega t} \sum_{n=-\infty}^{\infty} J_n(m_m) i^n e^{in\omega_m t}. \quad (4.7.3)$$

Using the properties of Bessel functions,

$$\begin{aligned} J_{-n}(m_m) &= (-1)^n J_n(m_m), \\ J_n(m_m) &\simeq \frac{1}{n!} \left(\frac{m_m}{2}\right)^n \quad (m_m \ll 1), \end{aligned} \quad (4.7.4)$$

and keeping terms up to first order in m_m , we obtain

$$\begin{aligned} E(t) &= E_0 \left\{ J_0(m_m) e^{i\omega t} + iJ_1(m_m) \left(e^{i(\omega+\omega_m)t} + e^{i(\omega-\omega_m)t} \right) \right\} \\ &= E_0 e^{i\omega t} + \frac{im_m}{2} E_0 \left(e^{i(\omega+\omega_m)t} - e^{i(\omega-\omega_m)t} \right) \\ &= E_0 e^{i\omega t} + E_1 e^{i(\omega+\omega_m)t} + E_{-1} e^{i(\omega-\omega_m)t}, \end{aligned} \quad (4.7.5)$$

where we have defined $E_{\pm 1} = \pm i(m_m/2)E_0$. As seen from Eq. (4.7.5), the EOM generates sidebands at frequencies $\omega \pm \omega_m$. The optical power P of this field is given by⁷

$$P = \frac{\hbar\omega}{2} \left(|E_0|^2 + |E_1|^2 + |E_{-1}|^2 + 2\text{Re}[(E_0^* E_1 + E_0 E_{-1}^*) e^{i\omega_m t}] + 2\text{Re}[E_1 E_{-1}^* e^{2i\omega_m t}] \right). \quad (4.7.6)$$

Among these terms, the component oscillating at ω_m contains a signal that is linear with respect to the mirror displacement. Demodulation is performed by mixing the detected signal with a local oscillator (LO) at the same modulation frequency ω_m . Depending on the LO phase, two types of demodulated signals are obtained: the in-phase (I-phase) component, corresponding to $\cos \omega_m t$, and the quadrature-phase (Q-phase) component, corresponding to $\sin \omega_m t$. After low-pass filtering to remove oscillatory terms, the error signals v_I and v_Q are obtained.

⁷where we have used identities such as $\text{Re}[E_0^* E_1 e^{i\omega_m t}] = \text{Re}[E_0 E_1^* e^{-i\omega_m t}]$.

We now calculate the PDH error signal used to control the cavity length L . The electric field reflected from the cavity E_r is given by

$$E_r = \frac{-r_1 + r_2 e^{-2i\phi}}{1 - r_1 r_2 e^{-2i\phi}} E_{\text{in}}, \quad (4.7.7)$$

where $\phi = \omega L/c$ is the phase accumulated over a half round trip. We define the complex amplitude reflectivity of the cavity by

$$r_c(\omega) \equiv \frac{-r_1 + r_2 e^{-2i\phi}}{1 - r_1 r_2 e^{-2i\phi}}, \quad (4.7.8)$$

so that

$$E_r \equiv r_c(\omega) E_{\text{in}}. \quad (4.7.9)$$

Substituting the modulated input field from Eq. (4.7.5), the reflected field becomes

$$E_{r,\text{mod}} = r_c(\omega) E_0 e^{i\omega t} + r_c(\omega + \omega_m) E_1 e^{i(\omega + \omega_m)t} + r_c(\omega - \omega_m) E_{-1} e^{i(\omega - \omega_m)t}. \quad (4.7.10)$$

The detected reflected power is then given by

$$\begin{aligned} P_{r,\text{mod}} = & P_{\text{DC}} + 2m_m |E_0|^2 \text{Re}[r_c^*(\omega) r_c(\omega + \omega_m) - r_c(\omega) r_c^*(\omega - \omega_m)] \cos \omega_m t \\ & + 2m_m |E_0|^2 \text{Im}[r_c^*(\omega) r_c(\omega + \omega_m) - r_c(\omega) r_c^*(\omega - \omega_m)] \sin \omega_m t + \mathcal{P}(2\omega_m), \end{aligned} \quad (4.7.11)$$

where P_{DC} is the DC component and $\mathcal{P}(2\omega_m)$ denotes terms oscillating at $2\omega_m$. After demodulation in the I- and Q-phases and removal of high-frequency components, the error signals are given by

$$\begin{aligned} v_{r,I} &= m_m |E_0|^2 \text{Re}[r_c^*(\omega) r_c(\omega + \omega_m) - r_c(\omega) r_c^*(\omega - \omega_m)], \\ v_{r,Q} &= m_m |E_0|^2 \text{Im}[r_c^*(\omega) r_c(\omega + \omega_m) - r_c(\omega) r_c^*(\omega - \omega_m)]. \end{aligned} \quad (4.7.12)$$

Figure 4.12 shows the normalized intracavity power and the PDH error signal.

4.7.2 Coherent control for squeezing angle

The coherent control scheme is a technique to stabilize the squeezing angle generated by an OPO as well as the homodyne readout angle [35, 36]. An alternative approach employs a phase-modulated seed field at the carrier frequency. However, even a small amount of seed light can couple classical laser noise into the system, limiting the squeezing level at low frequencies [37]. For this reason, the coherent control scheme is more commonly adopted in gravitational-wave detectors. A schematic diagram of the coherent control scheme is shown in Fig. 4.13. In the coherent control scheme, a coherent locking field (CLF) with a frequency offset Ω_{CLF} from the main laser frequency ω_0 is injected into the OPO cavity to generate two independent error signals for controlling the squeezing phase and the homodyne detection angle. The first error signal is obtained by demodulating the beat note between the CLF and the idler field generated in the OPO. This signal provides information about the relative phase between the CLF and the OPO pump field. The

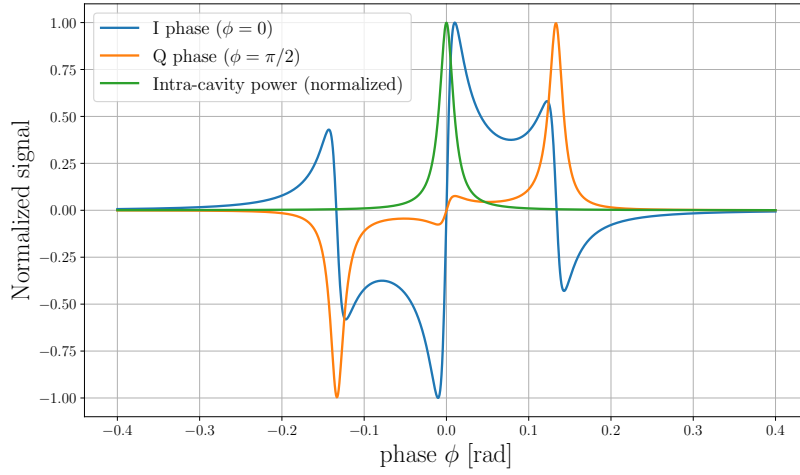


Figure 4.12: PDH error signal and normalized intracavity power. The cavity length is $L = 1$ m, mirror reflectivities are $r_1 = r_2 = \sqrt{0.99}$, the modulation index is $m_m = 0.15$ rad, the modulation frequency is $\omega_m = 20$ MHz, and the laser wavelength is $\lambda = 1064$ nm.

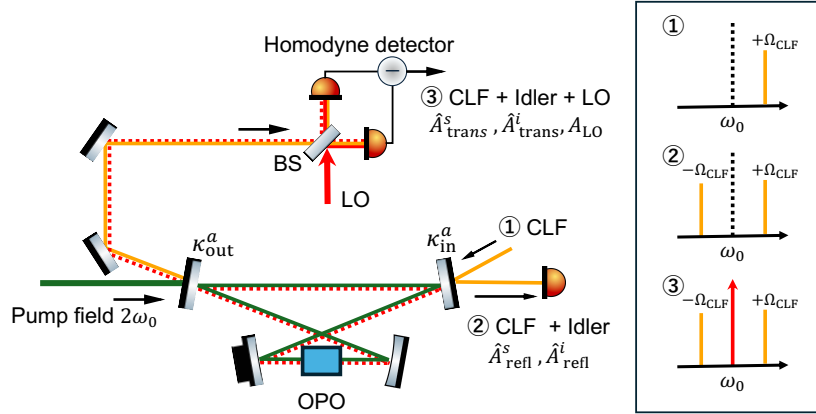


Figure 4.13: Left: Schematic overview of the optical setup. Right: Amplitudes of the optical field at three different locations.

second error signal is derived from the beat signal between the CLF transmitted through the OPO and the homodyne local oscillator (LO). This signal is used to control the relative phase between the OPO pump field and the LO. The CLF detuning is chosen such that the corresponding cavity detuning $|\Delta_s|$ remains within the cavity linewidth, i.e., $|\Delta_s| \leq \kappa$, so that the CLF is efficiently coupled into the cavity.

We consider the case where a pump field at $2\omega_0$ and a CLF at $\omega_0 + \Omega_{\text{CLF}}$ are injected into the OPO cavity. The intracavity pump amplitude is denoted by $|b|e^{i\theta_b}$. We denote the intracavity signal mode associated with the CLF by $\hat{a}_s$, and the generated idler mode by $\hat{a}_i$. With an incident CLF field $|A_s|e^{i\theta_{\text{CLF}}}$, the equations of motion are

$$\dot{\hat{a}}_s = -\kappa^a \hat{a}_s + i\Delta_{\text{cav}} \hat{a}_s + \frac{\varepsilon}{2} |b| e^{i\theta_b} \hat{a}_i^\dagger + \sqrt{2\kappa_{\text{in}}^a} |A_s| e^{i\theta_{\text{CLF}}}, \quad (4.7.13)$$

$$\dot{\hat{a}}_i = -\kappa^a \hat{a}_i + i\Delta_{\text{cav}} \hat{a}_i + \frac{\varepsilon}{2} |b| e^{i\theta_b} \hat{a}_s^\dagger. \quad (4.7.14)$$

Assuming that the cavity detuning of the CLF relative to the carrier, Δ_{cav} , is sufficiently

small compared with the cavity decay rate ($\kappa^a \gg \Delta_{\text{cav}}$), and considering the steady state ($\dot{\hat{a}}_s = \dot{\hat{a}}_i = 0$), the intracavity CLF and idler fields are obtained as

$$\hat{a}_s = \frac{\sqrt{2\kappa^a}|A_s|}{\kappa^a(1-x^2)}e^{i\theta_{\text{CLF}}} \quad (4.7.15)$$

$$\hat{a}_i = \frac{\sqrt{2\kappa^a}|A_s|}{\kappa^a(1-x^2)}xe^{i(\theta_b-\theta_{\text{CLF}})}, \quad (4.7.16)$$

where $x = \varepsilon|b|/2\kappa^a$ is the dimensionless nonlinear coupling parameter. The CLF reflected from the OPO is given by the input-output boundary condition as

$$\hat{A}_{\text{refl}}^s = \left(\frac{2\kappa_{\text{in}}^a}{\kappa^a(1-x^2)} - 1 \right) |A_s|e^{i\theta_{\text{CLF}}} \quad (4.7.17)$$

$$\hat{A}_{\text{refl}}^i = \frac{\sqrt{2\kappa_{\text{in}}^a}|A_s|x}{\kappa^a(1-x^2)}e^{i(\theta_b-\theta_{\text{CLF}})} \quad (4.7.18)$$

The measured reflected optical power is then given by

$$P_{\text{refl}} = \left| \hat{A}_{\text{refl}}^s e^{i(\omega_0+\Omega_{\text{CLF}})t} + \hat{A}_{\text{refl}}^i e^{i(\omega_0-\Omega_{\text{CLF}})t} \right|^2 \quad (4.7.19)$$

$$= |\hat{A}_{\text{refl}}^s|^2 + |\hat{A}_{\text{refl}}^i|^2 + 2\text{Re} \left[\hat{A}_{\text{refl}}^s \hat{A}_{\text{refl}}^{i\dagger} e^{2i\Omega_{\text{CLF}}t} \right] \quad (4.7.20)$$

$$= |\hat{A}_{\text{refl}}^s|^2 + |\hat{A}_{\text{refl}}^i|^2 + \frac{4|A_s|^2\kappa_{\text{in}}^ax}{\kappa^a(1-x^2)} \left(\frac{2\kappa_{\text{in}}^a}{\kappa^a(1-x^2)} - 1 \right) \cos(2\Omega_{\text{CLF}}t + 2\theta_{\text{CLF}} - \theta_b). \quad (4.7.21)$$

By demodulating this signal with $\sin(2\Omega_{\text{CLF}}t + \phi_{\text{mod}})$ and applying a low-pass filter, the DC component is removed. The resulting demodulated signal becomes

$$P_{\text{refl,mod}} = \frac{2|A_s|^2\kappa_{\text{in}}^ax}{\kappa^a(1-x^2)} \left(\frac{2\kappa_{\text{in}}^a}{\kappa^a(1-x^2)} - 1 \right) \sin(\phi_{\text{mod}} - 2\theta_{\text{CLF}} + \theta_b). \quad (4.7.22)$$

Thus, by appropriately choosing the demodulation phase ϕ_{mod} , a linear error signal can be obtained around the relative phase $2\theta_{\text{CLF}} - \theta_b$ between the pump field and the CLF.

Furthermore, the squeezing angle relative to the homodyne detection angle can be controlled using the beat signal between the CLF transmitted through the OPO and the homodyne LO. The transmitted CLF field from the OPO is given by

$$\hat{A}_{\text{trans}}^s = \frac{2\sqrt{\kappa_{\text{in}}^a\kappa_{\text{out}}^a}|A_s|}{\kappa^a(1-x^2)}e^{i\theta_{\text{CLF}}} \quad (4.7.23)$$

$$\hat{A}_{\text{trans}}^i = \frac{2\sqrt{\kappa_{\text{in}}^a\kappa_{\text{out}}^a}|A_s|x}{\kappa^a(1-x^2)}e^{i(\theta_b-\theta_{\text{CLF}})}. \quad (4.7.24)$$

If the LO field is written as $A_{\text{LO}}e^{i\omega_0t+i\xi_{\text{LO}}}$, the optical powers incident on the two photodiodes of the homodyne detector, P^{HD1} and P^{HD2} , are given by

$$P^{\text{HD1}} = \frac{1}{2} \left| \hat{A}_{\text{trans}}^s e^{i(\omega_0+\Omega_{\text{CLF}})t} + \hat{A}_{\text{trans}}^i e^{i(\omega_0-\Omega_{\text{CLF}})t} + A_{\text{LO}}e^{i\omega_0t+i\xi_{\text{LO}}} \right|^2, \quad (4.7.25)$$

$$P^{\text{HD2}} = \frac{1}{2} \left| \hat{A}_{\text{trans}}^s e^{i(\omega_0+\Omega_{\text{CLF}})t} + \hat{A}_{\text{trans}}^i e^{i(\omega_0-\Omega_{\text{CLF}})t} - A_{\text{LO}}e^{i\omega_0t+i\xi_{\text{LO}}} \right|^2. \quad (4.7.26)$$

Here we assume a 50/50 beam splitter in front of the homodyne detector, with amplitude reflectivity and transmissivity both equal to $1/\sqrt{2}$. A beat signal between the transmitted CLF and the LO appears at the frequency Ω_{CLF} . The resulting differential photocurrent is written as

$$P_{\text{diff}} = 4 \operatorname{Re} \left[\hat{A}_{\text{trans}}^s A_{\text{LO}}^\dagger e^{i(\Omega_{\text{CLF}} t - \xi_{\text{LO}})} + \hat{A}_{\text{trans}}^i A_{\text{LO}}^\dagger e^{-i(\Omega_{\text{CLF}} t + \xi_{\text{LO}})} \right] \quad (4.7.27)$$

$$= \frac{8\sqrt{\kappa_{\text{in}}^a \kappa_{\text{out}}^a} |A_s| |A_{\text{LO}}|}{\kappa^a (1 - x^2)} \left\{ \cos(\Omega_{\text{CLF}} t + \theta_{\text{CLF}} - \xi_{\text{LO}}) + x \cos(\Omega_{\text{CLF}} t - \theta_b + \theta_{\text{CLF}} + \xi_{\text{LO}}) \right\}. \quad (4.7.28)$$

The relative phase between the CLF and the pump field is assumed to be controlled such that the demodulation phase satisfies $\phi_{1,\text{mod}} = 2\theta_{\text{CLF}} - \theta_b$. Defining the squeezing angle relative to the homodyne local oscillator as

$$\theta_{\text{sqs}} = \frac{\theta_b}{2} - \xi_{\text{LO}}, \quad (4.7.29)$$

the phase component appearing in the differential signal demodulated at $\cos(\Omega_{\text{CLF}} t + \phi_{2,\text{mod}})$ becomes

$$P_{\text{diff,mod}} \propto \cos(\theta_{\text{CLF}} - \xi_{\text{LO}} - \phi_{2,\text{mod}}) + x \cos(\theta_{\text{CLF}} - \theta_b + \xi_{\text{LO}} - \phi_{2,\text{mod}}) \quad (4.7.30)$$

$$= \cos\left(\frac{\phi_{1,\text{mod}}}{2} - \phi_{2,\text{mod}} + \theta_{\text{sqs}}\right) + x \cos\left(\frac{\phi_{1,\text{mod}}}{2} - \phi_{2,\text{mod}} - \theta_{\text{sqs}}\right) \quad (4.7.31)$$

$$= \sqrt{1 + x^2 + 2x \cos(\phi_{1,\text{mod}} - 2\phi_{2,\text{mod}}) \cos(\theta_{\text{sqs}} + \zeta)}, \quad (4.7.32)$$

where the phase ζ is defined as

$$\zeta = \arctan \left[\frac{1 - x}{1 + x} \tan \left(\frac{\phi_{1,\text{mod}}}{2} - \phi_{2,\text{mod}} \right) \right]. \quad (4.7.33)$$

Therefore, the demodulated differential signal at the homodyne detector is given by

$$P_{\text{diff}} = \frac{4\sqrt{\kappa_{\text{in}}^a \kappa_{\text{out}}^a} |A_s| |A_{\text{LO}}|}{\kappa^a (1 - x^2)} \sqrt{1 + x^2 + 2x \cos(\phi_{1,\text{mod}} - 2\phi_{2,\text{mod}}) \cos(\theta_{\text{sqs}} + \zeta)}. \quad (4.7.34)$$

This expression shows that an error signal for controlling the squeezing angle can be obtained by adjusting the two demodulation phases.

Chapter 5

Principle of frequency dependent squeezing based on Quantum Teleportation

5.1 Motivation

This study aims to demonstrate the principle of an optimal injection scheme for frequency-dependent squeezed states tailored to the interferometer configurations planned for next-generation gravitational-wave detectors.

The next-generation detector proposed as the Einstein Telescope (ET) consists of three interferometers arranged in an equilateral triangular configuration. Each interferometer is composed of two dual-recycled Fabry–Pérot Michelson interferometers optimized for different frequency bands. Figure 5.1 shows a schematic overview of the ET optical configuration. Of the two interferometers, one is optimized for low-frequency detection (ET-LF), targeting the frequency range from 3 Hz to 30 Hz, while the other is optimized for high-frequency detection (ET-HF), covering the frequency range from 30 Hz to 10 kHz.

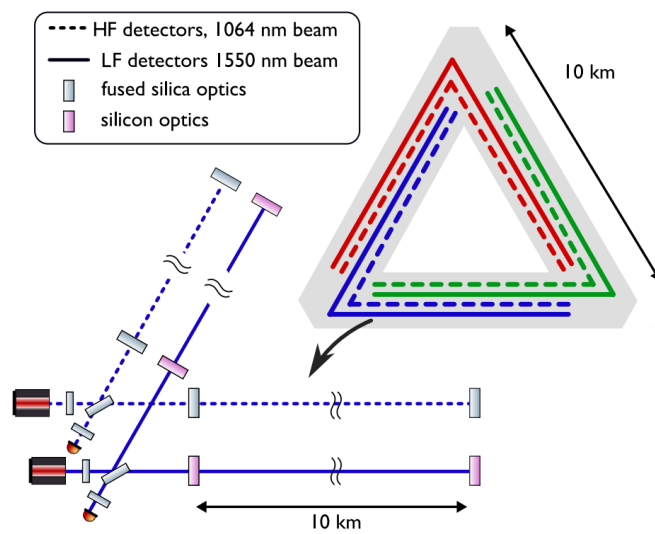


Figure 5.1: Schematic of the Einstein Telescope[24].

In ET-LF, a detuned resonant sideband extraction (detuned RSE) configuration is adopted in order to improve the sensitivity in the low-frequency band. In a detuned signal

recycling cavity, optomechanical effects such as the optical spring modify the quantum noise response of the interferometer in a frequency-dependent manner. Consequently, in order to optimally match the frequency-dependent squeezing angle of the injected squeezed state, two filter cavities are in principle required.

However, constructing and operating low-loss filter cavities on the kilometer scale poses significant technical and economic challenges. Motivated by this difficulty, comparative studies between the conventional filter-cavity approach and alternative methods have been carried out, primarily through numerical simulations. To date, three main approaches have been considered [38]:

- **A scheme based on two filter cavities**
- **A hybrid scheme combining a filter cavity with EPR entanglement**
- **A scheme based on quantum teleportation**

While experimental demonstrations of frequency-dependent squeezing using filter cavities or EPR entanglement have been reported for resonant (non-detuned) interferometer configurations, **no experimental demonstration or comparative study targeting frequency-dependent squeezing for detuned configurations has yet been reported.**

To identify the optimal approach for ET-LF, it is essential to experimentally investigate the characteristics and practical limitations of each method. Therefore, this study focuses on the most recent approach based on quantum teleportation and aims to present the world's first experimental demonstration of its underlying principle.

5.2 Overview of quantum-teleportation-based squeezing

Quantum teleportation is a protocol that enables the transfer of an unknown quantum state using shared entanglement and classical communication, without physically transmitting the state itself [39]. In continuous-variable quantum optics, this concept was theoretically extended to optical field quadratures by Braunstein and Kimble [40] and subsequently demonstrated experimentally by Furusawa *et al.* [41]. Based on this framework, Nishino *et al.* proposed a scheme to realize frequency-dependent squeezing suitable for a detuned interferometer using quantum teleportation [42]. The basic concept is closely related to the EPR squeezing scheme discussed in Sec. 3.5.2. In this approach, the interferometer itself effectively plays the role of a filter cavity, and frequency dependence of the squeezing angle is induced through quantum entanglement between different frequency modes. A schematic overview of the optical configuration for frequency-dependent squeezing via quantum teleportation (QT squeezing scheme) is shown in Fig. 5.2. The protocol can be divided into four main stages:

1. **Quantum state generation**
2. **Quadrature rotation by a detuned interferometer**
3. **Signal detection**

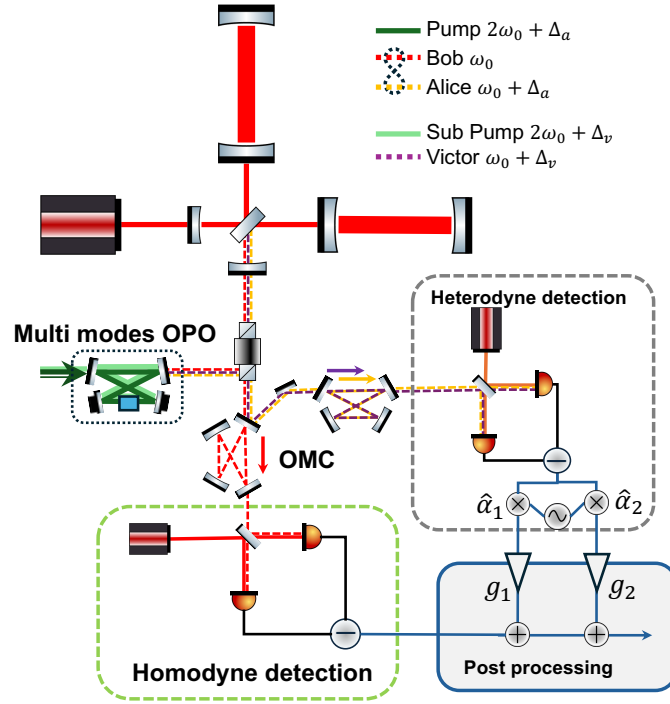


Figure 5.2: Optical layout of frequency-dependent squeezing based on quantum teleportation.

4. Post-processing

In this scheme, three quantum states, referred to as Bob, Alice, and Victor, are generated by a multimode OPO and injected into the dark port of the interferometer. Because these states occupy different frequency modes, they experience different interferometer responses. The interferometer outputs are separated by an output mode cleaner (OMC), which removes higher-order spatial modes and separates the frequency components. Signal detection is performed using homodyne and heterodyne detectors. By applying appropriate post-processing to the measurement results, an effective frequency-dependent squeezing is realized.

The details of each stage are described in the following sections, and the resulting quantum noise-limited strain sensitivity achievable with this method is derived.

5.3 Quantum state preparation

We define the carrier frequency of the laser interferometer as ω_0 , which is referred to as the Bob frequency mode. Using a single multimode OPO system, three quantum states at different frequencies are generated. The quantum state at frequency ω_0 is referred to as Bob, that at $\omega_0 + \Delta_a$ as Alice, and that at $\omega_0 + \Delta_v$ as Victor.

Uncorrelated vacuum fluctuations in the frequency modes corresponding to Alice, Bob, and Victor are denoted by $\delta\hat{X}^a$, $\delta\hat{X}^b$, and $\delta\hat{X}^v$, respectively. They are expressed in terms

of the amplitude and phase quadratures as

$$\delta\hat{\mathbf{X}}^a = \underbrace{[\delta\hat{X}_1^a(\Omega), \delta\hat{X}_2^a(\Omega)]^T}_{\omega_0 + \Delta_a}, \quad \delta\hat{\mathbf{X}}^b = \underbrace{[\delta\hat{X}_1^b(\Omega), \delta\hat{X}_2^b(\Omega)]^T}_{\omega_0}, \quad (5.3.1)$$

$$\delta\hat{\mathbf{X}}^v = \underbrace{[\delta\hat{X}_1^v(\Omega), \delta\hat{X}_2^v(\Omega)]^T}_{\omega_0 + \Delta_v}. \quad (5.3.2)$$

These quadratures satisfy the canonical commutation and variance relations

$$[\delta\hat{X}_1^a(\Omega), \delta\hat{X}_2^a(\Omega')] = 2i, \quad (5.3.3)$$

$$V(\delta\hat{X}_i^a(\Omega)) = 1 \quad (i = 1, 2), \quad (5.3.4)$$

and the same relations hold for the Bob and Victor modes.

The entangled pair of quantum states, Alice and Bob, is generated by a nondegenerate OPO. When the OPO is pumped at frequency $2\omega_0 + \Delta_a$, vacuum fluctuations in the signal mode at ω_0 and the idler mode at $\omega_0 + \Delta_a$ are converted into a two-mode squeezed state. Defining the quadratures of the generated Alice and Bob modes as

$$\hat{\mathbf{a}} = [\hat{a}_1(\Omega), \hat{a}_2(\Omega)]^T, \quad \hat{\mathbf{b}} = [\hat{b}_1(\Omega), \hat{b}_2(\Omega)]^T, \quad (5.3.5)$$

the input-output relation given in Eq. (4.4.13) shows that Alice and Bob form a quantum-correlated two-mode squeezed state, which can be expressed as

$$\begin{aligned} \begin{bmatrix} \hat{a}_1(\Omega) \\ \hat{a}_2(\Omega) \\ \hat{b}_1(\Omega) \\ \hat{b}_2(\Omega) \end{bmatrix} &= \underbrace{\begin{bmatrix} T_d & 0 & T_{\text{od}} \cos \theta_b & T_{\text{od}} \sin \theta_b \\ 0 & T_d & T_{\text{od}} \sin \theta_b & -T_{\text{od}} \cos \theta_b \\ T_{\text{od}} \cos \theta_b & T_{\text{od}} \sin \theta_b & T_d & 0 \\ T_{\text{od}} \sin \theta_b & -T_{\text{od}} \cos \theta_b & 0 & T_d \end{bmatrix}}_{\text{squeezing port}} \begin{bmatrix} \delta\hat{X}_1^a \\ \delta\hat{X}_2^a \\ \delta\hat{X}_1^b \\ \delta\hat{X}_2^b \end{bmatrix} \\ &+ \underbrace{\begin{bmatrix} T_d^l & 0 & T_{\text{od}}^l \cos \theta_b & T_{\text{od}}^l \sin \theta_b \\ 0 & T_d^l & T_{\text{od}}^l \sin \theta_b & -T_{\text{od}}^l \cos \theta_b \\ T_{\text{od}}^l \cos \theta_b & T_{\text{od}}^l \sin \theta_b & T_d^l & 0 \\ T_{\text{od}}^l \sin \theta_b & -T_{\text{od}}^l \cos \theta_b & 0 & T_d^l \end{bmatrix}}_{\text{loss port}} \begin{bmatrix} \delta\hat{X}_1^a \\ \delta\hat{X}_2^a \\ \delta\hat{X}_1^b \\ \delta\hat{X}_2^b \end{bmatrix}. \end{aligned} \quad (5.3.6)$$

Assuming that the decay rate κ , the escape efficiency η_{esc} , and the nonlinear gain factor x of the OPO are identical for all frequency modes, the matrix elements can be written as

$$T_d = -1 + \frac{2\eta_{\text{esc}}(1 + i\Omega/\kappa)}{(1 + i\Omega/\kappa)^2 - x^2}, \quad T_{\text{od}} = \frac{2\eta_{\text{esc}}x}{(1 + i\Omega/\kappa)^2 - x^2}, \quad (5.3.7)$$

$$T_d^l = \frac{2\sqrt{\eta_{\text{esc}}(1 - \eta_{\text{esc}})}(1 + i\Omega/\kappa)}{(1 + i\Omega/\kappa)^2 - x^2}, \quad T_{\text{od}}^l = \frac{2\sqrt{\eta_{\text{esc}}(1 - \eta_{\text{esc}})}x}{(1 + i\Omega/\kappa)^2 - x^2}. \quad (5.3.8)$$

Here, the first term represents the contribution from vacuum fields entering through the squeezing port, while the second term corresponds to vacuum fluctuations entering through optical loss channels, such as transmission through mirrors other than the input coupler.

Next, we describe the quantum state Victor, which is to be teleported. Victor is a

squeezed state generated by an OPO pumped at frequency $2\omega_0 + 2\Delta_v$. The quadratures of Victor at the output of the OPO are defined as

$$\hat{\mathbf{v}} = [\hat{v}_1, \hat{v}_2]^T. \quad (5.3.9)$$

Using the vacuum fluctuations $\delta\hat{\mathbf{X}}^v$ around the frequency $\omega_0 + \Delta_v$, the Victor quadratures can be written as

$$\hat{\mathbf{v}} = T_{\text{OPO}} \delta\hat{\mathbf{X}}^v + T_{\text{OPO}}^l \delta\hat{\mathbf{X}}_l^v. \quad (5.3.10)$$

The first term represents vacuum fluctuations entering through the squeezing port, which generate the squeezed state, while the second term corresponds to vacuum fields entering through loss channels associated with optical losses. The transfer matrices T_{OPO} and T_{OPO}^l are given by

$$\begin{aligned} T_{\text{OPO}} &= \begin{bmatrix} -1 + \frac{2\eta_{\text{esc}}(1+i\Omega/\kappa+x\cos\theta_b)}{1-x^2+\Omega/\kappa(2i-\Omega/\kappa)} & -\frac{2\eta_{\text{esc}}x\sin\theta_b}{1-x^2+\Omega/\kappa(2i-\Omega/\kappa)} \\ -\frac{2\eta_{\text{esc}}x\sin\theta_b}{1-x^2+\Omega/\kappa(2i-\Omega/\kappa)} & -1 + \frac{2\eta_{\text{esc}}(1+i\Omega/\kappa-x\cos\theta_b)}{1-x^2+\Omega/\kappa(2i-\Omega/\kappa)} \end{bmatrix}, \\ T_{\text{OPO}}^l &= \begin{bmatrix} \frac{2\sqrt{\eta_{\text{esc}}(1-\eta_{\text{esc}})}(1+i\Omega/\kappa+x\cos\theta_b)}{1-x^2+\Omega/\kappa(2i-\Omega/\kappa)} & \frac{2\sqrt{\eta_{\text{esc}}(1-\eta_{\text{esc}})}x\sin\theta_b}{1-x^2+\Omega/\kappa(2i-\Omega/\kappa)} \\ \frac{2\sqrt{\eta_{\text{esc}}(1-\eta_{\text{esc}})}x\sin\theta_b}{1-x^2+\Omega/\kappa(2i-\Omega/\kappa)} & \frac{2\sqrt{\eta_{\text{esc}}(1-\eta_{\text{esc}})}(1+i\Omega/\kappa-x\cos\theta_b)}{1-x^2+\Omega/\kappa(2i-\Omega/\kappa)} \end{bmatrix}. \end{aligned} \quad (5.3.11)$$

In conventional filter cavity-based squeezing injection schemes, only one frequency mode at the interferometer output is measured. In contrast, the present approach combines measurements of three frequency modes, namely $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{v}}$. This multimode measurement is a key feature that enables the reconstruction of an effective frequency-dependent squeezing through quantum teleportation.

5.4 Quadrature rotation by a detuned interferometer

The interferometer adopted in ET-LF is a dual-recycled Fabry–Pérot Michelson interferometer operated in a detuned resonant-sideband-extraction (RSE) configuration, in which the signal-recycling cavity is slightly detuned from resonance. In this section, we describe the quadrature response of the interferometer for each frequency mode.

Since the input field Bob, $\hat{\mathbf{b}}$, has the same frequency as the resonant mode of the arm cavities, it experiences the full detuned-RSE interferometer response, in which both the gravitational-wave signal and quantum fluctuations appear. The output field from the interferometer, $\hat{\mathbf{B}}$, is given by [43, 44]

$$\begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} = \frac{1}{M} \left(\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} + \tau \frac{\sqrt{2\mathcal{K}}}{h_{\text{SQL}}} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} h \right), \quad (5.4.1)$$

where

$$M = 1 + \rho^2 e^{4i\beta} - 2\rho e^{2i\beta} \left(\cos 2\phi + \frac{\mathcal{K}}{2} \sin 2\phi \right), \quad (5.4.2)$$

$$C_{11} = C_{22} = (1 + \rho^2) \left(\cos 2\phi + \frac{\mathcal{K}}{2} \sin 2\phi \right) - 2\rho \cos 2\beta, \quad (5.4.3)$$

$$C_{12} = -\tau^2 (\sin 2\phi + \mathcal{K} \sin^2 \phi), \quad (5.4.4)$$

$$C_{21} = \tau^2 (\sin 2\phi - \mathcal{K} \cos^2 \phi), \quad (5.4.5)$$

$$D_1 = -(1 + \rho e^{2i\beta}) \sin \phi, \quad (5.4.6)$$

$$D_2 = (1 - \rho e^{2i\beta}) \cos \phi. \quad (5.4.7)$$

Here, ρ and τ denote the amplitude reflectivity and transmissivity of the signal-recycling mirror, respectively, and ϕ represents the detuning of the signal-recycling cavity. The arm-cavity bandwidth γ is given by

$$\gamma = \frac{T_i c}{4L}, \quad (5.4.8)$$

where T_i is the power transmissivity of the input test mass. The phase delay β and the ponderomotive coupling factor $\mathcal{K}$ associated with the arm cavities are defined as

$$\beta \equiv \arctan\left(\frac{\Omega}{\gamma}\right), \quad (5.4.9)$$

$$\mathcal{K} \equiv \frac{8P_c \omega_0}{mcL\Omega^2(\gamma^2 + \Omega^2)}, \quad (5.4.10)$$

with P_c denoting the circulating power in the arm cavities.

In contrast, the interferometer exhibits a different response for the Alice and Victor modes. By appropriately choosing the interferometer parameters and the frequency offsets of Alice and Victor, the interferometer effectively acts as a filter cavity for these modes. As a result, the quadratures of Alice and Victor acquire frequency-dependent rotations.

The interferometer responses for Alice and Victor can be written as

$$\begin{aligned} \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} &= e^{i\beta_a} \begin{bmatrix} \cos \theta_a & -\sin \theta_a \\ \sin \theta_a & \cos \theta_a \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix}, \\ \begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \end{bmatrix} &= e^{i\beta_v} \begin{bmatrix} \cos \theta_v & -\sin \theta_v \\ \sin \theta_v & \cos \theta_v \end{bmatrix} \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix}, \end{aligned} \quad (5.4.11)$$

where θ_a and θ_v represent the quadrature rotation angles induced by the interferometer, and β_a and β_v denote the corresponding phase delays. In the present scheme, frequency-dependent squeezing is realized by teleporting the Alice and Victor states, which have undergone frequency-dependent quadrature rotations, onto the Bob mode.

5.4.1 Conditions on the frequency modes of Alice and Victor

The frequency dependence of the quadrature rotation experienced by a given frequency mode is determined by the detuning and the linewidth of the effective cavity. Therefore,

in order to obtain the desired frequency-dependent quadrature rotations for the Alice and Victor modes, their frequency offsets must simultaneously satisfy two conditions: one on the effective linewidth and the other on the effective detuning¹. Here, following Ref. [45], we model the interferometer as an effective cavity formed by the signal-recycling mirror and the input test mass (ITM), with the end test mass (ETM) acting as the end mirror.

As shown in Eq. (3.4.23), the effective linewidth of the interferometer for the Alice and Victor modes is given by

$$\gamma_{a,v} = \frac{\tau^2}{1 + \rho^2 + 2\rho \cos 2\phi_{a,v}} \gamma, \quad (5.4.12)$$

where γ is the arm-cavity bandwidth. Accordingly, the quadrature rotation angle $\phi_{a,v}$ experienced by the Alice and Victor modes inside the interferometer is given by

$$\phi_{a,v} = \frac{1}{2} \arccos\left(\frac{\tau^2 (\gamma/\gamma_{v,a}) - \rho^2 - 1}{2\rho}\right) + n\pi. \quad (5.4.13)$$

Since the phase rotation $\phi_{a,v}$ depends on the frequency offsets of Alice and Victor as

$$\phi_{a,v} = \phi + \frac{\Delta_{a,v} L_{\text{SEC}}}{c}, \quad (5.4.14)$$

the condition that relates the effective linewidth and the frequency offset can be written as

$$\Delta_{a,v} = \frac{c}{L_{\text{SEC}}} \left(\frac{1}{2} \arccos\left(\frac{\tau^2 (\gamma/\gamma_{v,a}) - \rho^2 - 1}{2\rho}\right) + n_{a,v}\pi - \phi \right). \quad (5.4.15)$$

In addition, the effective detuning of the interferometer for the Alice and Victor frequency modes is given by

$$\delta_{a,v} = \delta_{a,v}^{\text{arm}} + \frac{2\rho \sin 2\phi_{a,v}}{1 + \rho^2 + 2\rho \cos 2\phi_{a,v}} \gamma, \quad (5.4.16)$$

where the first term represents the detuning of the Alice and Victor frequencies from the arm-cavity resonance, $\delta_{a,v}^{\text{arm}} = \text{Mod}(\Delta_{a,v}/\omega_{\text{FSR}}^{\text{arm}})$, and the second term describes the contribution of the signal-extraction cavity (SEC).

Equations (5.4.15) and (5.4.16) together define the conditions that the Alice and Victor frequency modes must satisfy. Using the ET-LF design parameters listed in Table 5.1, we obtain $\Delta_a \simeq 319$ MHz and $\Delta_v \simeq 962$ MHz. In summary, the Alice and Victor frequency modes must be chosen such that the interferometer provides the desired effective linewidth and detuning, allowing it to act as a frequency-dependent quadrature rotator for these auxiliary modes.

¹When the states are generated by an OPO, the Alice and Victor modes must also be resonant with Bob in the OPO cavity, which requires their frequency offsets to be integer multiples of the OPO free spectral range.

Definition	Symbol	Value
Carrier wavelength	λ	1550 nm
ITM power transmittance	T_i	7000 ppm
Signal-extraction mirror power transmittance	τ^2	90%
Mirror mass	m	211 kg
Power at beam splitter	I_0	63 W
Detuning of the SEC	ϕ_0	0.75 rad
Arm initial length	$L_{\text{arm}}^{(0)}$	10 km
SEC initial length	$L_{\text{SEC}}^{(0)}$	100 m
Arm-cavity bandwidth	γ_{arm}	8.35 Hz
Bandwidth/detuning of the first filter cavity	γ_1/δ_1	4.27/19.54 Hz
Bandwidth/detuning of the second filter cavity	γ_2/δ_2	1.64/− 7.62 Hz
Arm length tuning	δL_{arm}	30 000 λ
SEC length tuning	δL_{SEC}	14 030 λ
Detuning of Alice	Δ_a	769 kHz + 213 FSR _{SEC}
Detuning of Victor	Δ_v	656 kHz + 642 FSR _{SEC}

Table 5.1: Parameters of the ETLF.

5.5 Signal detection

After the frequency separation by the output mode cleaner (OMC), the Bob mode is measured by a homodyne detector, while the Alice and Victor modes are measured by a heterodyne detector. This asymmetric detection scheme reflects the different roles of the three modes: Bob carries the gravitational-wave signal, whereas Alice and Victor are auxiliary modes used to construct frequency-dependent squeezing via quantum teleportation.

5.5.1 Signal measurement by homodyne detection

When the Bob output field $\hat{\mathbf{B}}$ from the interferometer is measured with a homodyne angle ζ , the detected quadrature is given by Eq. (4.5.9) as

$$\hat{B}_{\text{out}} = \mathbf{H}_{\zeta}^{\text{Hom}} \hat{\mathbf{B}} = \hat{B}_1 \cos \zeta + \hat{B}_2 \sin \zeta. \quad (5.5.1)$$

In this subsection, we briefly summarize the measured quadrature that enters the final noise spectrum.

5.5.2 Bell measurement by heterodyne detection

To extract quantum correlations between the Alice and Victor modes, we perform a joint measurement using heterodyne detection. The quadratures of Alice and Victor at the interferometer output are combined into a single vector,

$$\hat{\mathcal{A}} \equiv [\hat{V}_1, \hat{V}_2, \hat{A}_1, \hat{A}_2]^T. \quad (5.5.2)$$

When the Alice and Victor modes are measured with a heterodyne angle ξ and demodulated into the in-phase (I) and quadrature-phase (Q) components, the corresponding

measurement vectors acting on the quadratures are given by

$$\begin{aligned} \mathbf{H}_{\xi,I}^{\text{Het}} &= \frac{1}{\sqrt{2}} [\cos \xi, \sin \xi, \cos \xi, \sin \xi], \\ \mathbf{H}_{\xi,Q}^{\text{Het}} &= \frac{1}{\sqrt{2}} \left[\cos\left(\xi - \frac{\pi}{2}\right), -\sin\left(\xi - \frac{\pi}{2}\right), \cos\left(\xi + \frac{\pi}{2}\right), -\sin\left(\xi + \frac{\pi}{2}\right) \right]. \end{aligned} \quad (5.5.3)$$

Accordingly, the measured quadratures are

$$\hat{\boldsymbol{\alpha}} = [\alpha_1, \alpha_2]^T = [\mathbf{H}_{\xi,Q}^{\text{Het}} \hat{\mathbf{A}}, \mathbf{H}_{\xi,I}^{\text{Het}} \hat{\mathbf{A}}]^T. \quad (5.5.4)$$

For the specific choice $\xi = \pi/2$, Eq. (5.4.11) yields

$$\begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{V}_1 - \hat{A}_1 \\ \hat{V}_2 + \hat{A}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \hat{v}_1 \cos \theta_v - \hat{v}_2 \sin \theta_v - e^{i\beta_a} (\hat{a}_1 \cos \theta_a - \hat{a}_2 \sin \theta_a) \\ \hat{v}_1 \sin \theta_v + \hat{v}_2 \cos \theta_v + e^{i\beta_a} (\hat{a}_1 \sin \theta_a + \hat{a}_2 \cos \theta_a) \end{bmatrix}, \quad (5.5.5)$$

which explicitly corresponds to the sum and difference of the Alice and Victor quadratures. Since these observables are joint measurements spanning two distinct quantum systems, this detection scheme constitutes a Bell measurement. Using the rotation matrix $R(\theta)$, Eq. (5.5.5) can be rewritten as

$$\sqrt{2} \hat{\boldsymbol{\alpha}} = R(\theta_v) \hat{\mathbf{v}} + R(-\theta_a) \hat{\mathbf{a}}, \quad (5.5.6)$$

The Bell measurement outcomes obtained in Eq. (5.5.5) are subsequently used to reduce the output noise of the Bob mode through classical feedforward, which realizes quantum teleportation of the squeezed state. This procedure is discussed in detail in the following section.

5.6 Post-processing

In this section, we show how quantum teleportation is completed by applying classical linear post-processing to the outcomes of the Bell measurement. The purpose of this procedure is to minimize the output noise variance at Bob by optimally combining the measurement results.

5.6.1 Frequency-dependent squeezing via quantum teleportation

We first clarify which quadrature is effectively measured at Bob after the post-processing. We choose the squeezing angle $\theta_b = \pi$, such that amplitude squeezing is generated. In this configuration, Victor is amplitude-squeezed and the fluctuation of $\hat{v}_1$ is reduced. At the same time, quantum correlations between Alice and Bob lead to the simultaneous squeezing of the quadratures $\hat{a}_1 + \hat{b}_1$ and $\hat{a}_2 - \hat{b}_2$. We collect these squeezed combinations into the vector

$$\hat{\mathbf{x}}^{\text{sqz}} \equiv [\hat{b}_1 + \hat{a}_1, \hat{b}_2 - \hat{a}_2]^T. \quad (5.6.1)$$

Using $\hat{\mathbf{x}}^{\text{sqz}}$ and the Bell measurement outcome $\hat{\boldsymbol{\alpha}}$, the interferometer response in Eq. (5.4.1) can be rewritten as

$$\begin{aligned} \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} &= \mathcal{H}_b \hat{\mathbf{x}}^{\text{sqz}} + \mathcal{H}_b R(\theta_a) \sqrt{2} \hat{\boldsymbol{\alpha}} - \mathcal{H}_b R(\theta_a + \theta_v) \hat{\mathbf{v}} \\ &\quad + \frac{\tau}{M} \frac{\sqrt{2\mathcal{K}}}{h_{\text{SQL}}} \begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \end{bmatrix} h, \end{aligned} \quad (5.6.2)$$

where

$$\mathcal{H}_b = \frac{1}{M} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}. \quad (5.6.3)$$

The first term in Eq. (5.6.2) corresponds to the quadratures squeezed by quantum entanglement and approaches zero as the squeezing level increases. The second term represents the Bell measurement outcomes, which are accessible experimentally and can be removed by classical post-processing. In the ideal limit of infinite squeezing ($r \rightarrow \infty$) and in the absence of loss, the Bob output reduces to

$$\hat{\mathbf{B}} \rightarrow -\mathcal{H}_b R(\theta_a + \theta_v) \hat{\mathbf{v}} + \frac{\tau}{M} \frac{\sqrt{2\mathcal{K}}}{h_{\text{SQL}}} \begin{bmatrix} \hat{D}_1 \\ \hat{D}_2 \end{bmatrix} h. \quad (5.6.4)$$

The angles $\theta_a(\Omega)$ and $\theta_v(\Omega)$ depend on the interferometer parameters and on the frequency modes of Alice and Victor, and therefore exhibit frequency dependence. By choosing these rotations to cancel the quadrature rotation induced by $\mathcal{H}_b$, frequency-dependent squeezing can be realized.

As an example, we consider homodyne detection with $\zeta = \pi/2$, corresponding to the measurement of the phase quadrature $\hat{B}_2$. In this case,

$$\begin{aligned} \hat{B}_2 &\rightarrow -\frac{1}{M} \{C_{21} \cos(\theta_a + \theta_v) + C_{22} \sin(\theta_a + \theta_v)\} \hat{v}_1 \\ &\quad - \frac{1}{M} \{-C_{21} \sin(\theta_a + \theta_v) + C_{22} \cos(\theta_a + \theta_v)\} \hat{v}_2 + \frac{\tau}{M} \frac{\sqrt{2\mathcal{K}}}{h_{\text{SQL}}} D_2 h. \end{aligned} \quad (5.6.5)$$

To suppress the contribution from the anti-squeezed quadrature $\hat{v}_2$ and retain only the squeezed fluctuation $\hat{v}_1$, the condition

$$-C_{21} \cos(\theta_a + \theta_v) + C_{22} \sin(\theta_a + \theta_v) = 0 \quad (5.6.6)$$

must be satisfied, which leads to the frequency-dependent phase condition

$$\theta_a(\Omega) + \theta_v(\Omega) = -\arctan\left(\frac{C_{22}}{C_{21}}\right). \quad (5.6.7)$$

Under this condition, the frequency-dependent squeezed state generated at Victor is faithfully reproduced at Bob, enabling broadband quantum noise reduction. This completes the realization of frequency-dependent squeezing via quantum teleportation. Figure 5.3 shows the quadrature rotations of each mode calculated using the parameters listed in Table 5.1, where the quadrature rotation at Bob is defined as $\theta_b \equiv -\arctan(C_{22}/C_{21})$.

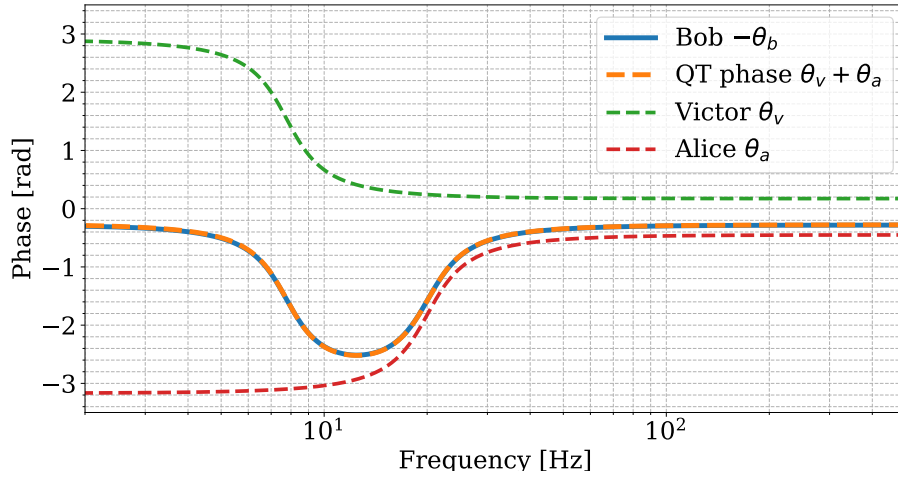


Figure 5.3: Frequency-dependent quadrature rotations for Bob, Alice, and Victor.

5.6.2 Optimal noise reduction using a Wiener filter

We now evaluate the noise variance obtained by frequency-dependent squeezing via quantum teleportation using realistic parameters, and derive the optimal linear filtering. We consider homodyne detection with $\zeta = \pi/2$, corresponding to the measurement of the phase quadrature $\hat{B}_2$. The measured signal can be written as

$$\begin{aligned}\hat{B}_2 &= \Gamma e^{i\beta_b} \left(\hat{b}_1 \cos \theta_b - \hat{b}_2 \sin \theta_b \right) + \tau e^{i(\beta+\beta_s)} \frac{\sqrt{2\mathcal{K}}}{h_{\text{SQL}}} D_2 h \\ &\equiv \hat{B}_2^{\text{QN}} + \hat{B}_2^{\text{GW}},\end{aligned}\quad (5.6.8)$$

where we have defined

$$\Gamma = \frac{\sqrt{C_{21}^2 + C_{22}^2}}{|M|}, \quad \beta_b = \arg(M^*), \quad \theta_b = -\arctan\left(\frac{C_{22}}{C_{21}}\right). \quad (5.6.9)$$

Next, the homodyne output of Bob is linearly combined with the Bell measurement outcomes using frequency-dependent gains g_1 and g_2 ,

$$\hat{B}_2^{\text{tel}} = \hat{B}_2^{\text{QN}} - g_1 \hat{\alpha}_1 - g_2 \hat{\alpha}_2, \quad (5.6.10)$$

and the noise variance of the filtered signal is minimized. Such a linear combination that minimizes the residual noise variance is known as a Wiener filter. The resulting noise variance is given by

$$\begin{aligned}V(\hat{B}_2^{\text{tel}}) &= V(\hat{B}_2^{\text{QN}}) + |g_1|^2 V(\hat{\alpha}_1) + |g_2|^2 V(\hat{\alpha}_2) \\ &\quad - g_1^* V(\hat{B}_2^{\text{QN}}, \hat{\alpha}_1) - g_1 V(\hat{\alpha}_1, \hat{B}_2^{\text{QN}}) \\ &\quad - g_2^* V(\hat{B}_2^{\text{QN}}, \hat{\alpha}_2) - g_2 V(\hat{\alpha}_2, \hat{B}_2^{\text{QN}}) \\ &\quad - g_1 g_2^* V(\hat{\alpha}_1, \hat{\alpha}_2) - g_1^* g_2 V(\hat{\alpha}_2, \hat{\alpha}_1).\end{aligned}\quad (5.6.11)$$

Assuming that the optimal frequency-dependent rotation condition $\theta_b + \theta_a = -\theta_v$ is satisfied, the individual covariance terms reduce to

$$V(\hat{B}_2^{\text{QN}}) = \Gamma^2 V_d, \quad (5.6.12)$$

$$V(\hat{\alpha}_1) = \frac{1}{2} (V_d + V_{\text{sqz}} \cos^2 \theta_v + V_{\text{anti-sqz}} \sin^2 \theta_v), \quad (5.6.13)$$

$$V(\hat{\alpha}_2) = \frac{1}{2} (V_d + V_{\text{sqz}} \sin^2 \theta_v + V_{\text{anti-sqz}} \cos^2 \theta_v), \quad (5.6.14)$$

$$V(\hat{B}_2^{\text{QN}}, \hat{\alpha}_1) = \frac{1}{\sqrt{2}} e^{i(\beta_b - \beta_a)} \Gamma V_{od} \cos \theta_v, \quad (5.6.15)$$

$$V(\hat{B}_2^{\text{QN}}, \hat{\alpha}_2) = \frac{1}{\sqrt{2}} e^{i(\beta_b - \beta_a)} \Gamma V_{od} \sin \theta_v, \quad (5.6.16)$$

$$V(\hat{\alpha}_1, \hat{\alpha}_2) = \frac{1}{2} (V_{\text{sqz}} - V_{\text{anti-sqz}}) \cos \theta_v \sin \theta_v. \quad (5.6.17)$$

The optimal filter gains g_1 and g_2 are obtained by solving

$$\frac{\partial V(\hat{B}_2^{\text{tel}})}{\partial g_1} = 0, \quad \frac{\partial V(\hat{B}_2^{\text{tel}})}{\partial g_2} = 0, \quad (5.6.18)$$

which yields [46]

$$g_1^{\text{opt}} = \frac{V(\hat{B}_2^{\text{QN}}, \hat{\alpha}_1) V(\hat{\alpha}_2) - V(\hat{B}_2^{\text{QN}}, \hat{\alpha}_2) V(\hat{\alpha}_1, \hat{\alpha}_1)}{V(\hat{\alpha}_1) V(\hat{\alpha}_2) - |V(\hat{\alpha}_1, \hat{\alpha}_2)|^2}, \quad (5.6.19)$$

$$g_2^{\text{opt}} = \frac{V(\hat{B}_2^{\text{QN}}, \hat{\alpha}_2) V(\hat{\alpha}_1) - V(\hat{B}_2^{\text{QN}}, \hat{\alpha}_1) V(\hat{\alpha}_1, \hat{\alpha}_2)}{V(\hat{\alpha}_1) V(\hat{\alpha}_2) - |V(\hat{\alpha}_1, \hat{\alpha}_2)|^2}. \quad (5.6.20)$$

The minimized quantum noise variance, $S_{\min}^{\text{QN}} = \min[V(\hat{B}_2^{\text{tel}})]$, is then given by

$$S_{\min}^{\text{QN}} = |\Gamma|^2 \left(V_d - \frac{V_{od}^2}{V_d + V_{\text{sqz}}} \right). \quad (5.6.21)$$

In the high-squeezing limit $r \gg 1$, using $V_d \pm V_{od} = V_{\text{anti-sqz}}, V_{\text{sqz}}$, this expression reduces to

$$S_{\min}^{\text{QN}} \rightarrow 3|\Gamma|^2 V_{\text{sqz}}. \quad (5.6.22)$$

Compared with frequency-dependent squeezing using filter cavities (FC scheme), the quantum teleportation scheme (QT scheme) yields a noise variance that is larger by a factor of three. This originates from the fact that the FC scheme measures a single frequency mode generated by a degenerate OPO, whereas the QT scheme combines fluctuations from three independent frequency modes. As a result, the achievable squeezing level is degraded by $10 \log_{10} 3 \simeq 4.8$ dB.

The corresponding strain sensitivity is given by

$$S_h(\Omega) = \frac{3h_{\text{SQL}}^2}{2\mathcal{K}\tau^2} \frac{C_{21}^2 + C_{22}^2}{|D_2|^2} V_{\text{sqz}}. \quad (5.6.23)$$

Figure 5.4 shows the strain sensitivity curves obtained with frequency-dependent squeezing based on a filter cavity and with the quantum teleportation (QT) squeezing scheme,

assuming an ideal lossless case. Even in the absence of loss, the QT scheme suffers from a fundamental penalty of 4.8 dB. As a result, for the same squeezing level, the filter cavity scheme achieves better sensitivity over most of the frequency band.

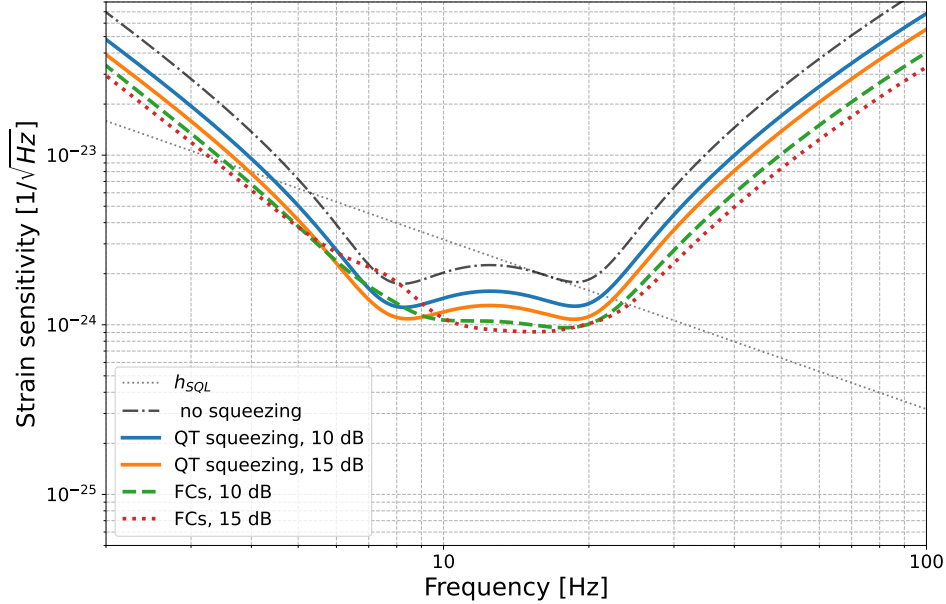


Figure 5.4: Comparison of strain sensitivity curves for the Filter Cavity scheme and the quantum teleportation (QT) squeezing scheme.

Another drawback of the QT squeezing scheme is that optical losses arising from mode mismatch, injection loss due to Faraday isolators, internal interferometer losses, and detector readout losses contribute independently to each frequency mode. As a consequence, their impact on the total noise is effectively tripled.

On the other hand, a key advantage of the QT squeezing scheme is that it does not require a physical filter cavity. For underground detectors such as the Einstein Telescope (ET), the construction of long-baseline filter cavities would require additional infrastructure, including vacuum chambers and underground caverns, leading to significant cost and complexity. Moreover, compared to a newly installed filter cavity, the main interferometer with 10 km arm cavities exhibits lower loss per unit length and therefore provides superior filtering performance.

In addition, length fluctuations of a filter cavity introduce frequency-dependent phase noise, making its control particularly challenging [47]. Since the interferometer is stabilized by multi-stage suspension systems, the QT scheme is less susceptible to frequency-dependent phase noise than the filter cavity scheme. This is the reason why the QT scheme achieves better sensitivity around 8 Hz.

Chapter 6

Experiment of frequency dependent squeezing based on Quantum Teleportation

In this chapter, the experimental setup required to demonstrate frequency-dependent squeezing based on quantum teleportation was constructed and its characteristics were investigated. In particular, an Output mode cleaner (OMC) and test cavities (TCs) were introduced, and their performances were evaluated through detailed characterization. Furthermore, after implementing homodyne detectors, squeezed arch measurements were carried out to assess the overall optical loss of the system. In this experiment, the frequency-dependent properties of optical cavities are used in place of a laser interferometer. Regarding the frequency notation, the frequencies of Bob, Alice, and Victor are set to $\omega_0 - \Lambda$, $\omega_0 + \Lambda$, and ω_0 , respectively, while the pump field frequency is set to $2\omega_0$. By appropriately tuning the frequencies of the main laser and the pump field, this configuration can be made consistent with the theoretical framework described in the previous section. Therefore, this difference in frequency assignment does not pose a fundamental issue for verifying the present technique.

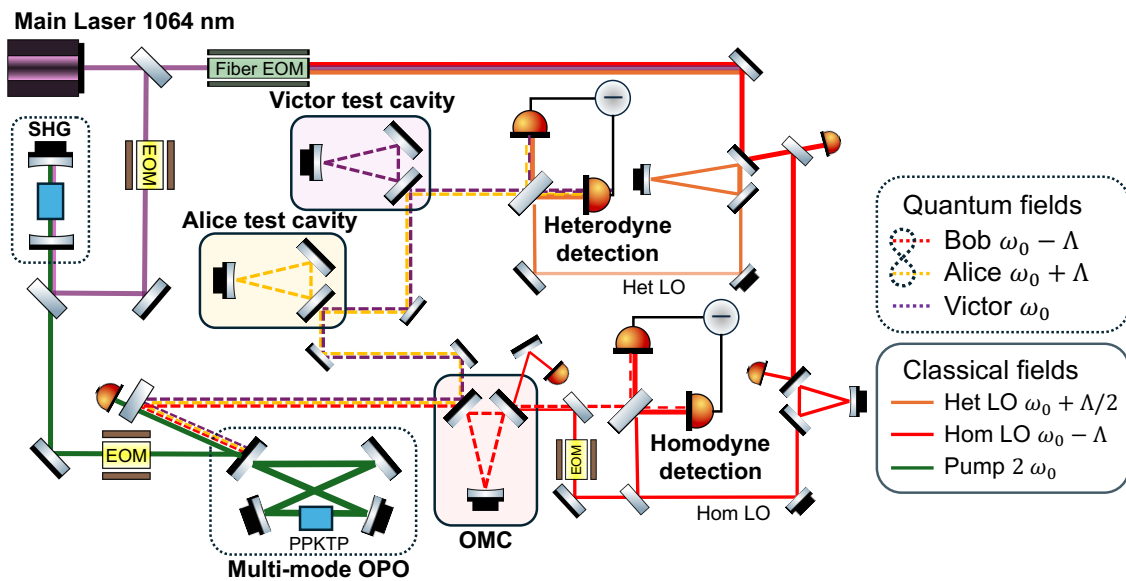


Figure 6.1: Schematic of the quantum teleportation squeezing experiment.

6.1 Experimental Overview

In this study, three quantum states referred to as Alice, Bob, and Victor are generated using a single pump field in a multi-mode optical parametric oscillator (OPO). These three states are frequency-separated by an optical cavity, referred to as the output mode cleaner (OMC). Bob's frequency mode is transmitted through the OMC and measured using homodyne detection, while the frequency modes of Alice and Victor are reflected by the OMC and subsequently injected into test cavities. The two test cavities are slightly detuned from resonance for the frequencies of Alice and Victor, respectively, and impart frequency-dependent quadrature rotations. Due to the frequency separation, Alice's mode does not resonate in Victor's test cavity and is almost entirely reflected. By independently controlling the resonance conditions of the two test cavities, different frequency dependencies of the squeezing angles can be applied to Alice and Victor. Alice and Victor are measured using heterodyne detection to perform a Bell measurement. Quantum teleportation is realized by applying classical post-processing to the Bell measurement results and Bob's measurement outcome, resulting in a noise variance with frequency-dependent squeezing.

6.1.1 Characteristics of the Verification Approach

In order to experimentally verify frequency-dependent squeezing based on quantum teleportation in a tabletop setup, this study adopts the following two approaches.

Quantum Measurements in the MHz frequency band

We generate squeezed vacuum light over a broad frequency range from approximately 10 kHz to 10 MHz, where technical noise is suppressed. In the low-frequency region, laser noise and thermal noise often exceed the optical shot noise, requiring advanced laser stabilization and vacuum infrastructure. Therefore, this study focuses on the high-frequency band where shot noise is dominant and measures frequency-dependent squeezing. Such high-frequency noise measurements are commonly adopted during the technology verification stage.

Test Cavities for squeezing angle rotation

Instead of a full-scale laser interferometer, this study employs two test cavities. This approach simplifies the overall control requirements while enabling the generation of Bob, Alice, and Victor using a single-pump OPO system and facilitating the experimental verification of quantum-teleportation-based squeezing. The test cavities are detuned from resonance and provide phase rotations equivalent to the frequency response experienced by squeezed vacuum light inside an interferometer. The cavity linewidths and detunings are chosen to be compatible with the OPO bandwidth and the measurement bandwidth of the photodetectors.

6.2 Experimental system

6.2.1 SHG

Second-harmonic generation (SHG) is performed using lithium niobate. SHG of a 1064 nm laser generates 532 nm light, which is used as the pump light for the OPO system. SHG cavity is stabilised by controlling the cavity length via the PDH method, where 12 MHz phase modulation is applied using a resonant electro-optic modulator (EOM). To maximize the conversion efficiency, the crystal temperature is maintained at approximately 103 °C. The conversion efficiency achieved is approximately 40 %, which is sufficient to generate pump power exceeding the OPO threshold (83 mW). Therefore, this is adequate for the present experiment, in which the OPO is operated in the sub-threshold regime.

6.2.2 Local oscillators

The local oscillators used in this experiment are generated from the sidebands produced by an EOSPACE fiber electro-optic modulator. The fiber EOM is driven at $\Lambda/2 = 424.8482$ MHz with an amplitude of 160 mV, generating sidebands up to the second order. The upper first-order modulation sideband, at $\omega_0 + \Lambda/2$, serves as the local oscillator for the heterodyne detection scheme, whereas the lower second-order modulation sideband, at $\omega_0 - \Lambda$, is utilized as the local oscillator for the homodyne detection. In addition, by applying phase modulation $\omega_m = 9$ MHz to the drive signal of the fiber EOM, PDH error signals can be generated for each sideband. The frequency distributions of the modes used in this study are shown in Fig. 6.2. The light emitted from the main laser passes through

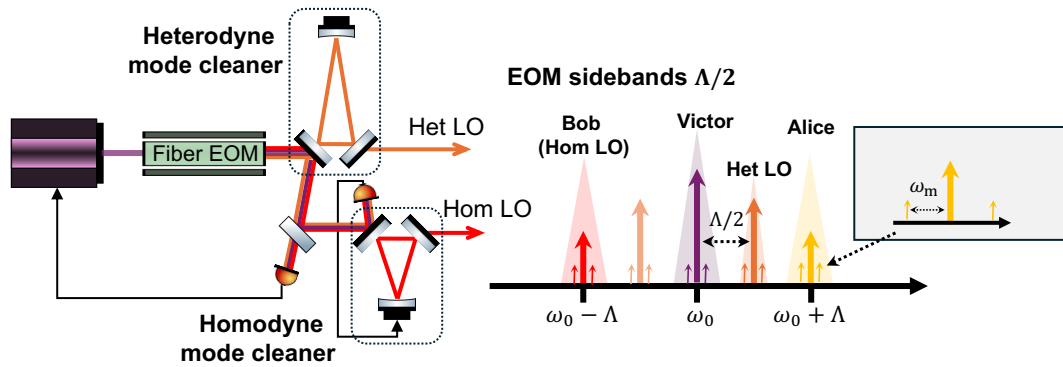


Figure 6.2: (a) Simplified schematic of the local oscillator preparation system. (b) Mode spacing diagram.

the fiber EOM, where sidebands are generated. The output from the fiber coupler is then injected into the heterodyne mode cleaner. The light reflected from the big mode cleaner is subsequently directed to the homodyne mode cleaner.

Heterodyne mode cleaner

The heterodyne mode cleaner is an optical cavity that transmits the frequency mode used as the local oscillator for heterodyne detection while reflecting the other modes. The cavity parameters are summarised in Table 6.1 . The reflected light is demodulated and

then fed back to the piezo actuator used to control the laser frequency, thereby maintaining cavity resonance.

Parameter	Value
Cavity length	0.84 m
Free spectral range	357 MHz
FWHM (S-pol)	108 kHz
Finesse (S-pol)	6614
Cavity waist	525 μm

Table 6.1: Summary of Heterodyne mode cleaner parameters.

Homodyne mode cleaner

The homodyne mode cleaner is an optical cavity that transmits the frequency mode used as the local oscillator for homodyne detection. The cavity parameters are summarized in Table 6.2. The reflected light is demodulated and sent to a piezo-mounted mirror, enabling active control of the cavity length.

Parameter	Value
Cavity length	0.21 m
Free spectral range	1430 MHz
FWHM (S-pol)	40 MHz
Finesse (S-pol)	35
Cavity waist (measured)	193 μm

Table 6.2: Summary of Homodyne mode cleaner parameters.

6.2.3 OPO system

In this study, we use an OPO system to generate three quantum states: the entangled modes Bob and Alice, and the squeezed state Victor. The parameters of the OPO system are summarized in Table 6.3. Fig. 6.3 illustrates the frequency distribution of Bob, Alice,

Parameter	Value
Input coupler power reflectivity at 1064 nm	83.9%
Input coupler power reflectivity at 532 nm	72.2%
Curved mirrors' radius of curvature	-38 mm
Round-trip optical path length	0.279 m
Escape efficiency (η_{esc})	$98 \pm 1 \%$
Linewidth at 1064 nm ($\text{FWHM}_{\text{OPO}}/2$)	12.1 MHz
Free spectral range (FSR_{OPO})	849.696 MHz
Finesse at 1064 nm	35.1
Threshold power	83 mW

Table 6.3: Summary of OPO parameters.

and Victor relative to the OPO resonance conditions. The nonlinear optical crystal used in the OPO is a PPKTP, which is mounted on a copper holder together with a temperature

control system based on a Peltier element. One end of the PPKTP crystal is wedge-polished to enable simultaneous resonance of the pump and fundamental fields at multiple positions within the cavity. The bow-tie cavity geometry prevents back reflections, in contrast to a Fabry–Pérot configuration.

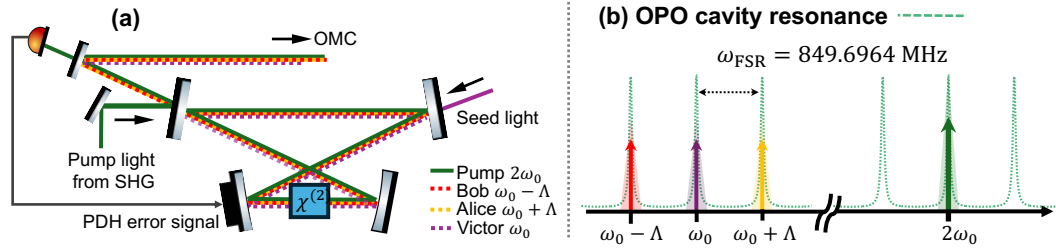


Figure 6.3: (a) Schematic of the OPO system. (b) Relation between the FSR of the OPO and the frequency modes.

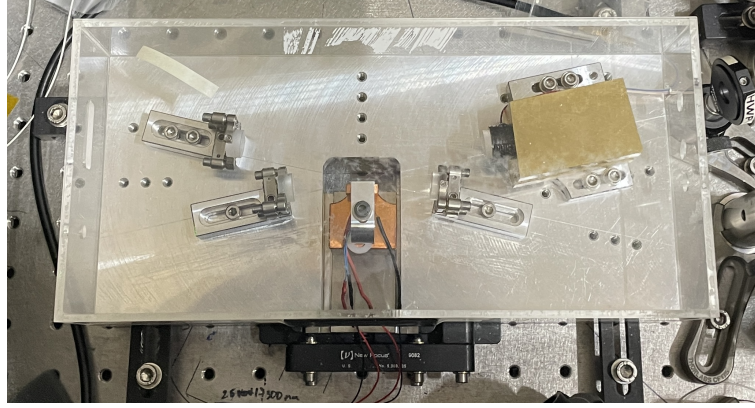


Figure 6.4: Photograph of the OPO setup.

To maximize the nonlinear optical efficiency, the crystal temperature is maintained at 32.4°C . Since the optimal temperature slightly depends on the incident pump power, it is fine-tuned for each measurement. The cavity length is controlled using the PDH method by imprinting sidebands onto the pump beam via an EOM. Both the pump and the seed beams are injected into the OPO system. The seed beam has half the frequency of the pump beam and matches the frequency of Victor. It is used for optimizing the crystal temperature and for aligning the OMC and the test cavity. For squeezing measurements, the seed light is not injected to generate a squeezed vacuum state and to suppress classical intensity and frequency noise.

6.2.4 Output mode cleaner

The Output Mode Cleaner (OMC) is an optical cavity that transmits only Bob's frequency mode among the quantum states generated by the OPO. The parameters of the OMC are summarized in Table 6.4. The free spectral range (FSR) of the OMC is chosen to be

$$\text{FSR}_{\text{OMC}} \sim 1.5 \text{FSR}_{\text{OPO}} = 1274 \text{ MHz}$$

so that Bob's mode is transmitted while the other modes are reflected. The cavity half-linewidth is set sufficiently larger than the squeezing bandwidth of 12.1 MHz. Furthermore, the input and end mirrors are designed to achieve critical coupling, ensuring maximum transmission of the resonant mode. In addition, a curved mirror with a radius of curvature

Parameter	Designed Value
Cavity length	0.2354 m
Free spectral range (FSR_{OMC})	1274 MHz
FWHM (S-pol)	43 MHz
Finesse (S-pol)	29.6

Table 6.4: Summary of OMC parameters.

of $R = 0.25$ m was used in the cavity. For this configuration, the cavity stability parameter is $g = 0.54$. The corresponding beam waist size is $w_0 = 205 \mu\text{m}$.

6.2.5 Test cavities

The test cavity (TC) is an optical cavity used to reproduce the squeezing-angle rotation that occurs in gravitational-wave interferometers. After Alice and Victor are reflected by the OMC, they subsequently pass through this detuned test cavity. The parameters of the TC are summarized in Table 6.4.

Parameter	Design value
Length	0.2971 m
Free spectral range (FSR_{TC})	1009.6964 MHz
FWHM (S-pol/P-pol)	1.6 MHz / 5.6 MHz
Finesse (S-pol/P-pol)	631 / 180

Table 6.5: Designed parameters for the test cavity.

In this experiment, two identical triangular cavities are employed. Each cavity is designed to be low-loss and over-coupled in order to minimize excess optical loss while ensuring a well-defined frequency response. The cavities are detuned with respect to their corresponding frequency modes. The Alice test cavity is detuned for the Alice frequency mode, such that the Victor mode is largely reflected. Similarly, the Victor test cavity is detuned for the Victor frequency mode, causing the Alice mode to be reflected. The cavity linewidth was set to be ≤ 1 MHz to ensure that the rotation of the squeezing angle occurs within the squeezing generation bandwidth of 12.1 MHz. In addition, the free spectral range (FSR) was chosen to satisfy the following relation in order to simplify the control of the cavity detuning:

$$\text{FSR}_{\text{TC}}/2 = \Lambda/2 + f_{\text{AOM}}. \quad (6.2.1)$$

Here, Λ denotes the sideband frequency generated by the fiber electro-optic modulator (EOM), and f_{AOM} represents the frequency shift produced by the first-order diffraction of the acousto-optic modulator (AOM). Further details of the detuning control scheme are described in subsection 6.3.4. A curved mirror with a radius of curvature of $R = 0.25$ m was used in the cavity. For this configuration, the cavity stability parameter is $g = 0.4$, corresponding to a beam waist size of $w_0 = 204 \mu\text{m}$.

6.2.6 Homodyne detector

The homodyne detector is used to measure Bob transmitted through the OMC. The homodyne detector employed in this experiment was developed by the research group of Prof. Ray-Kuang Lee at National Tsing Hua University (NTHU). The detector was designed by Dr. Chien-Ming Wu and fabricated by Dr. Te-Hwei Suen. The parameters characterizing the detector are listed in Table 6.6. The detector operates with ± 8 V nominal power supplies, within a range of ± 8 V to ± 15 V. The DC channel has a cutoff frequency of 20 kHz, and the AC channel provides a small-signal bandwidth¹ of 22 MHz. It covers the squeezing bandwidth generated by the OPO, which spans from a few kHz up to 12.1 MHz.

Parameters	Value
Detector type	InGaAs photodiodes
Wavelength	1064 nm
Photodiode Quantum efficiency η_{QE}	≥ 95 %
Anti-Reflection coating	R < 0.4 %
Angle of Incidence	20 degree
Optical input power	≤ 17 mW (each PD)
Photodiode active diameter	500 μ m
Capacity	14 pF at -2.5 V , 10.5 pF at -5 V
Dark current	98.0 and 98.3 pA at -5 V
Clearance at 2.5 MHz	30 dB with 30 mW input
Transimpedance gain	50 kV/A

Table 6.6: Performance of the Homodyne detector.

6.2.7 Heterodyne detector

The heterodyne detectors measure the sum and difference of the quadratures of the Alice and Victor fields. The parameters of the heterodyne detectors are summarized in Table 6.7.

Parameters	Value
Detector bandwidth	1 GHz
Photodetector NEP	0.74 mW
RF trans-impedance gain (R_T)	1000 V/A
Readout buffer gain	1.9
Balanced detector contrast	97 ± 1 %
Photodiode quantum efficiency	90 ± 2 %

Table 6.7: Performance of the Heterodyne detector.

¹The frequency at which the gain for small input signals drops by 3 dB. Within this range the amplification remains linear, while beyond this range the gain gradually rolls off.

6.3 Experiment

6.3.1 Measurement of the OPO Nonlinear Gain

The nonlinear gain provides a direct measure of the achievable squeezing level. In this measurement, a weak seed beam at the same frequency as Victor was injected into the OPO, and the amplification of the seed light was measured while stabilizing the OPO cavity length. The measurement was performed with the crystal temperature set to 32.4 °C, at which both the pump field and the seed field corresponding to Victor's frequency mode were simultaneously resonant with the OPO cavity, thereby maximizing the optical parametric amplification (OPA) effect. The nonlinear gain $\mathcal{G}$ was defined as the ratio of the output seed power to the input seed power. As shown in Eq. (4.3.1), it is given by

$$\mathcal{G} = \frac{1 + x^2 + 2x \cos \theta_b}{(1 - x^2)^2}, \quad (6.3.1)$$

where $x = |g|/\kappa_a = \sqrt{P/P_{\text{th}}}$ is the normalized pump parameter, P is the pump power, and P_{th} is the OPO oscillation threshold. Figure 6.5 shows the measured nonlinear gain as a function of the pump power P . By fitting the measured data with the theoretical model, the OPO oscillation threshold was estimated to be approximately $P_{\text{th}} \simeq 83$ mW.

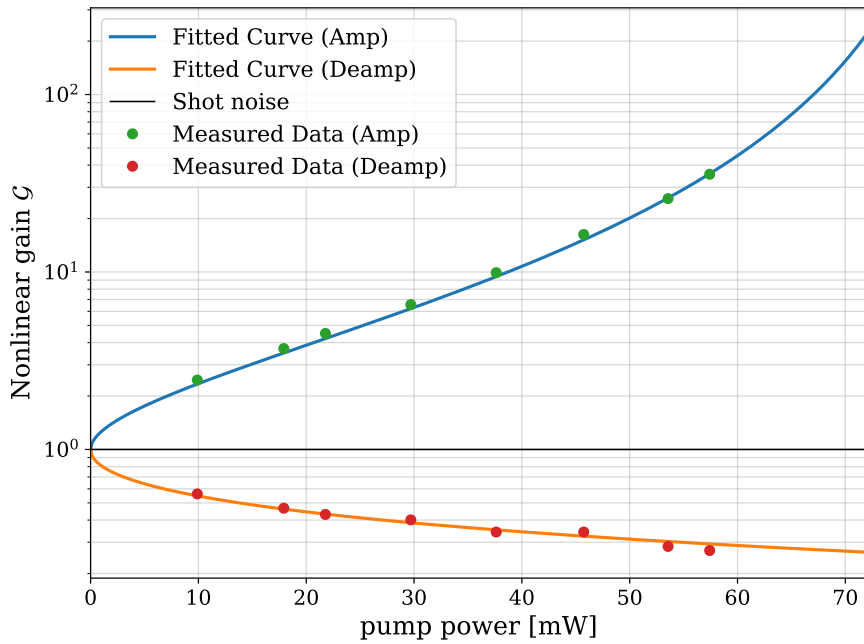


Figure 6.5: Nonlinear gain measurement at different pump power. The red dots represent the measured data. The blue line shows the amplification curve, and the orange line shows the de-amplification.

6.3.2 Mode-matching calculation

To minimize optical losses, mode matching between the propagation mode of the laser beam and the spatial eigenmode of the optical cavity is essential. Mode matching is

achieved by appropriately adjusting the beam size and waist position of the laser beam using lenses so that it matches the fundamental transverse mode of the cavity. In this study, the design and optimization of mode matching were performed using the optical simulation software JamMT. First, the initial beam parameters, namely the beam radius and waist position, were measured using a beam profiler. The measured data were fitted using the Gaussian beam propagation formula,

$$w(z) = w_0 \sqrt{1 + \frac{\lambda^2(z - z_0)^2}{\pi^2 w_0^4}}, \quad (6.3.2)$$

where λ is the laser wavelength, w_0 is the beam waist radius, and z_0 denotes the waist position. Subsequently, the mode-match assistant in JamMT was used to specify the focal lengths, thicknesses, and refractive indices of the lenses. By setting the desired beam waist size, the optimal lens configuration and positions were calculated.

6.3.3 Characterization of the OMC

Figure 6.6 shows a photograph of the OMC introduced in the experiment. The mirrors used in the OMC are summarized in Table 6.8.² After introducing the OMC into the experimental setup, several parameters were measured in order to evaluate its frequency separation characteristics and its impact on squeezing measurements. In particular, the finesse, FSR, FWHM, mode-matching efficiency, and optical power transmission were measured.

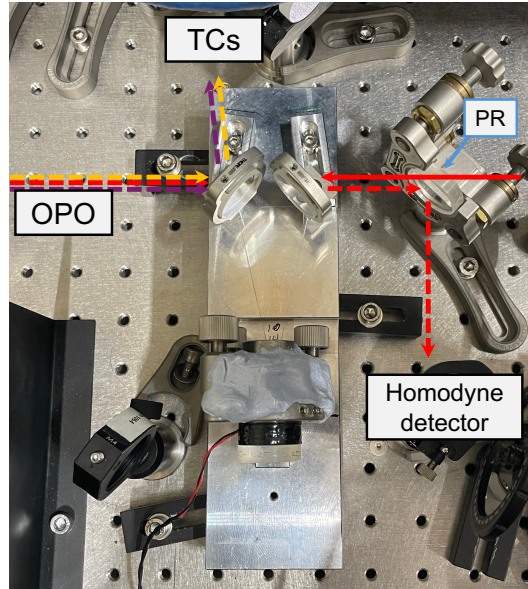


Figure 6.6: Image of the installed OMC. The input coupler is on the right and the output coupler on the left. A piezo-actuated mirror is mounted at the bottom. Blu-Tack is attached to damp mechanical resonances of the piezo-mounted mirror and mount.

²Reflectivity denotes the power reflectivity, AOI the angle of incidence, and ROC the radius of curvature.

Name	Manufacturer	Reflectivity	AOI	ROC
Input/Output Coupler	LASEROPTIK	90%	42 deg	Flat
Curved mirror	CVI SMCC-1025-0.25 UV HR	99.998 %	6 deg	250 mm

Table 6.8: Specifications of the mirrors composing the OMC.

Finesse, FSR, FWHM of the OMC

The finesse, FSR, and FWHM were measured to characterize the cavity performance. For the FSR measurement, the stroke of the PZT was insufficient to scan over one full FSR. Therefore, sidebands generated by a fiber EOM were used as a frequency reference. The cavity length was scanned using the PZT while the frequency of the second-order EOM sidebands was varied, and the FSR was determined when the resonance of a second-order sideband overlapped with that of the neighbouring second-order sideband separated by one FSR. In this configuration, the FSR corresponds to four times the EOM drive frequency.

The FWHM was obtained from the time-series data of the transmitted signal during the PZT scan. The time axis was converted to the frequency axis using the frequency separation between the TEM₀₀ mode and the EOM sideband. The finesse was then calculated from the measured FSR and FWHM. These parameters are summarised in Table 6.9. The uncertainty of the FSR is dominated by the frequency resolution of the EOM drive (1 MHz step), resulting in an estimated uncertainty of ± 4 MHz. The uncertainty of the FWHM was evaluated from repeated measurements, and the finesse uncertainty was obtained by error propagation.

Parameter	Designed Value	Measured Value
Cavity length	0.2354 m	0.229 m
Free spectral range (FSR _{OMC})	1274.5 MHz	1312 ± 4 MHz
FWHM (S-pol)	43 MHz	44 ± 3 MHz
Finesse (S-pol)	29.6	29.8 ± 2.0

Table 6.9: Summary of OMC parameters.

OMC mode-matching

The mode-matching efficiency, which is one of the optical losses that degrade the squeezing level, was experimentally evaluated. It was determined from the ratio of the TEM₀₀ resonance peak in the transmitted signal to the sum of all resonance peaks observed while scanning the OMC cavity length using a PZT actuator. The mode-matching efficiency is therefore given by

$$\eta_{\text{mm}} = \frac{\text{TEM}_{00}}{\text{TEM}_{00} + \sum_{i,j} \text{TEM}_{ij}}. \quad (6.3.3)$$

The measurement results are shown in Fig. 6.7. The mode-matching efficiency was estimated to be approximately 95 %. As mentioned earlier, since the PZT was unable to scan the cavity length over one full free spectral range (FSR), this measurement may underestimate the true mode-matching efficiency.

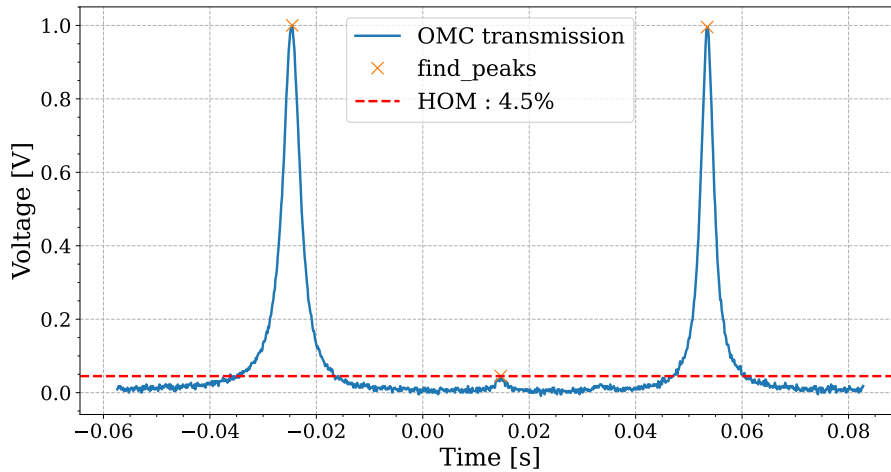


Figure 6.7: Normalized OMC transmission from the squeezing path with the cavity length scanned.

OMC control

The OMC is stabilized by injecting this control beam in the reverse propagation direction through a partial reflector with 99 % power reflectivity as shown in Fig. 6.8. The OMC control beam is an s-polarized beam obtained by picking off a portion of the homodyne LO and applying phase modulation with an EOM. The reflected beam provides the PDH error signal used to stabilize the cavity length. An image of the system is shown in Fig. 6.6.

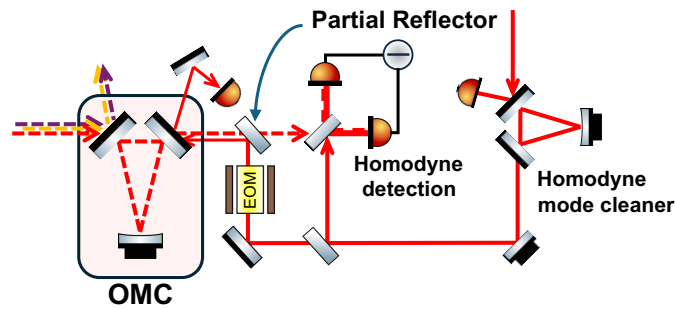


Figure 6.8: Schematic of the OMC control system.

Figure 6.9 shows the transmitted optical power of the OMC during cavity control. From 0 s to 6.5 s, the cavity length is scanned using a PZT mirror, and resonance peaks are observed. The apparent variation in the peak heights over time is primarily due to the limited sampling rate of the oscilloscope relative to the PZT scan speed, which results from the long measurement time. At approximately 6.5 s, feedback is applied using the PDH error signal, and the cavity is observed to be stably locked near the resonance point.

OMC Transmission

To evaluate the impact of the OMC on the measurement, the transmission of the OMC was measured with the OMC locked. The transmission was determined by comparing

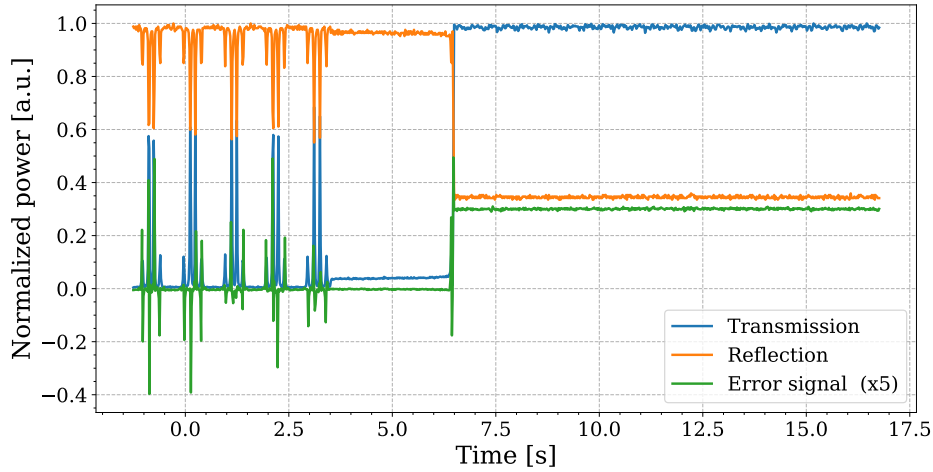


Figure 6.9: Transmitted control field's power of the OMC under cavity length control.

the optical power incident on the OMC with the power measured at the beam splitter located before the homodyne detector. From this measurement, the OMC transmission was found to be $84 \pm 1\%$. The transmission loss can be attributed to three main factors. First, approximately $7 \pm 1\%$ of the incident light is reflected due to mode mismatch and polarization mismatch. Second, a loss of about $5 \pm 1\%$ arises from the partial reflector (PR mirror) used to inject the OMC control beam. Finally, losses inside the cavity and along the optical propagation path were estimated to be approximately 3%. A summary of these loss contributions is provided in Table 6.10.

Parameter	Measured Value
Transmission	$84 \pm 1\%$
Reflection	$7 \pm 1\%$
PR mirror loss	$5 \pm 1\%$
internal loss	$3 \pm 1\%$

Table 6.10: Summary of OMC losses.

From the above measurements, we evaluate the OMC reflectivity for the Alice and Victor modes, $R_{a,v}$, when the OMC is controlled to transmit the Bob mode. Using Eq. (4.1.18), the reflectivity is given by

$$R_{a,v} = 1 - \frac{\kappa_b^2}{\kappa_b^2 + (\Delta_{\text{OMC}}^{a,v})^2} T_{\text{OMC}}. \quad (6.3.4)$$

Here, κ_b denotes the decay rate of the OMC and is given by

$$\kappa_b = \frac{\text{FWHM}}{2} = 6.25 \text{ MHz}. \quad (6.3.5)$$

The detunings of the Alice and Victor modes from the OMC resonance are denoted by Δ_{OMC}^a and Δ_{OMC}^v , respectively. When the OMC is locked to the Bob frequency mode,

these detunings are given by

$$\Delta_{\text{OMC}}^a = \min_{n \in \mathbb{Z}} |2\Lambda - n \text{FSR}_{\text{OMC}}| \simeq 387 \text{ MHz}, \quad (6.3.6)$$

$$\Delta_{\text{OMC}}^v = \min_{n \in \mathbb{Z}} |2\Lambda - n \text{FSR}_{\text{OMC}}| \simeq 462 \text{ MHz}. \quad (6.3.7)$$

Using these values, the reflectivities are calculated as

$$R_a \simeq 0.9997, \quad (6.3.8)$$

$$R_v \simeq 0.9998, \quad (6.3.9)$$

indicating that both the Alice and Victor modes are almost completely reflected by the OMC.

6.3.4 Characterization of the Test cavities

In this section, two test cavities were implemented in the experimental setup. Key cavity parameters, including the finesse, FSR, and FWHM were measured for each cavity. The cavity lengths were stabilized using the PDH locking technique.

As shown in Fig. 6.10, each test cavity adopts a mechanically rigid design in which the mirrors are mounted on a spacer, fixing the spatial geometry of the cavity. The specifications of the mirrors composing the test cavities are summarized in Table 6.11.

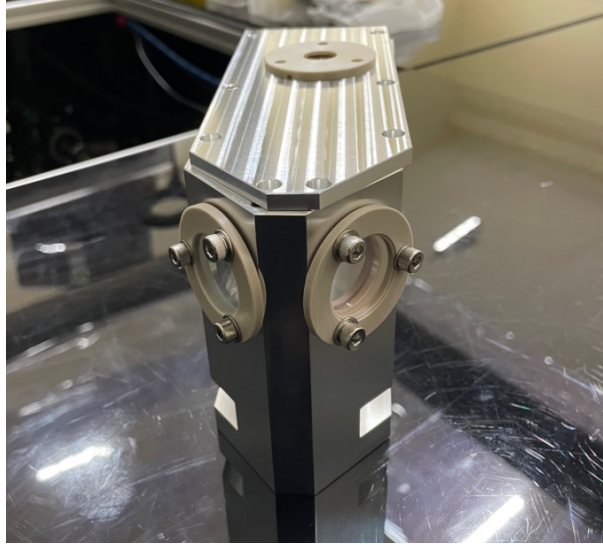


Figure 6.10: Image of the installed TCs.

Name	Manufacturer	Reflectivity	AOI	ROC
Input Coupler(AR coated)	Thorlab BB1-E03P	99.5%	42 deg	Flat
Output Coupler	LASEROPTIK	99.99%	42 deg	Flat
Rear curved mirror	FIVENINE	99.998 %	6 deg	250 mm

Table 6.11: Specifications of the mirrors composing the TCs.

Finesse, FSR, and FWHM of the test cavities

The optical properties of the test cavities were characterized using the control fields extracted from the heterodyne mode cleaner, as shown in Fig. 6.14. This approach was adopted because the seed light transmitted through the OPO is very weak, and no EOM is installed in the squeezed-light path in order to avoid additional optical losses.

First, frequency calibration of the time-series data was performed using 10 MHz sidebands generated by an EOM. While scanning the cavity length with a PZT, the transmitted and reflected signals for p -polarized light were measured, as shown in Fig. 6.11. The highest peak corresponding to the TEM_{00} mode appears around 0.0012 s, while sideband peaks separated by 10 MHz are observed around -0.002 s and 0.005 s. Since the temporal separation between the TEM_{00} peak and the sideband peaks corresponds to 10 MHz, this provides a reliable frequency calibration. Figure 6.12 shows the mode scans for the

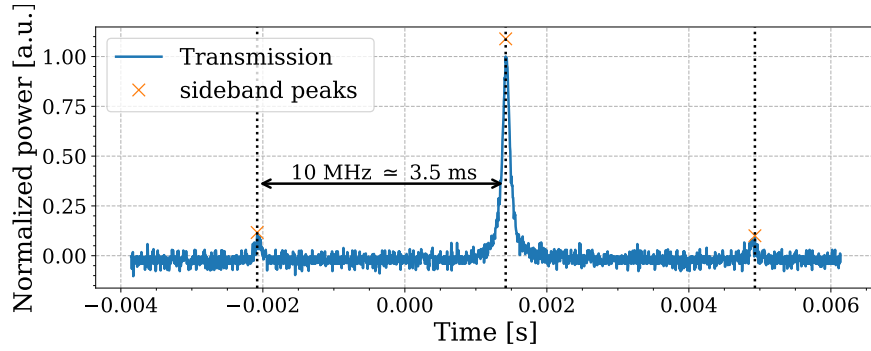


Figure 6.11: Resonance peaks of the test cavity including the modulation sidebands.

s - and p -polarized fields after frequency calibration. By fitting the transmitted signals with Lorentzian functions, the full widths at half maximum (FWHMs) were found to be 0.17 ± 0.01 MHz and 0.34 ± 0.01 MHz for the s and p polarizations, respectively. Figure 6.13 shows the mode scan obtained by sweeping the cavity length over more than

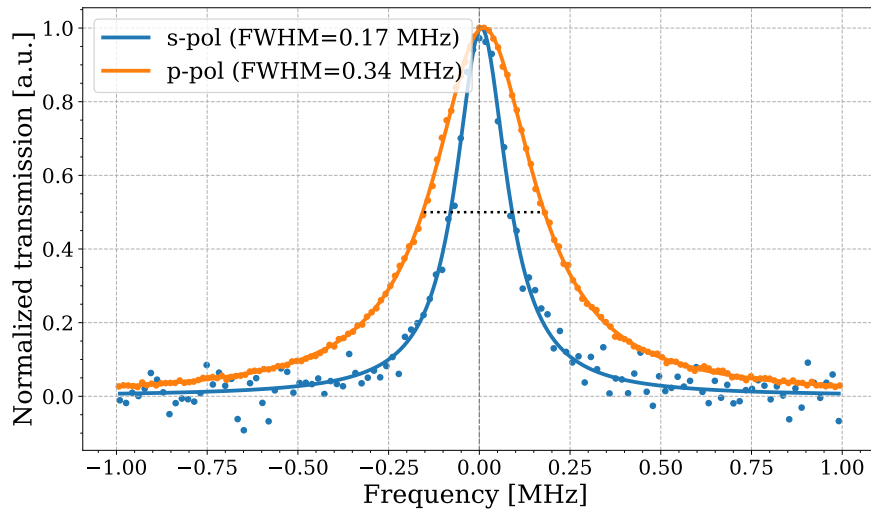


Figure 6.12: Mode scan of the test cavity for s - and p -polarized light.

one free spectral range (FSR). From this measurement, the FSR of the test cavity was found to be 1069 MHz. Accordingly, the finesse of the test cavity for each polarization is given by

$$\mathcal{F}_s \simeq (6.3 \pm 0.4) \times 10^3, \quad (6.3.10)$$

$$\mathcal{F}_p \simeq (3.1 \pm 0.1) \times 10^3 \quad (6.3.11)$$

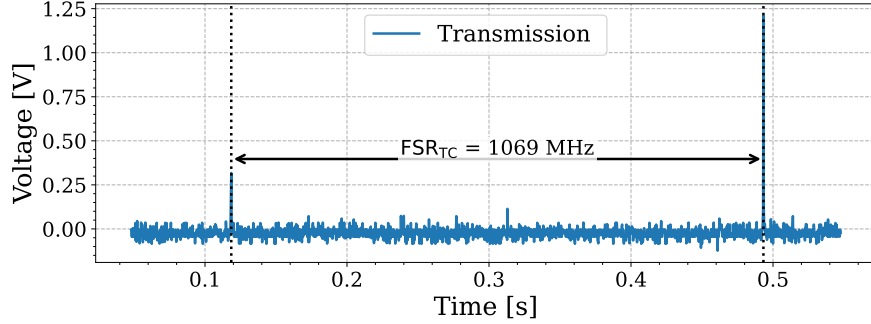


Figure 6.13: Mode scan over more than one FSR for p -polarized light.

The measured cavity parameters are summarized in Table 6.12. The results suggest that the reflectivity of the input coupler is higher than the design value, resulting in a narrower linewidth. While a narrower linewidth imposes stricter requirements on detuning control, an excessively broad linewidth would cause the quadrature rotation induced by quantum teleportation to exceed the squeezing bandwidth. Therefore, careful optimization of the cavity linewidth is required for stable operation.

Parameter	Design value	Measured value
Length	0.2971 m	0.2804 m
Free spectral range (FSR_{TC})	1009.6964 MHz	1069 MHz
FWHM (S-pol/P-pol)	1.6 MHz / 5.6 MHz	0.17 ± 0.01 MHz / 0.34 ± 0.01 MHz
Finesse (S-pol/P-pol)	631 / 180	$(6.3 \pm 0.4) \times 10^3$ / $(3.1 \pm 0.1) \times 10^3$

Table 6.12: Measured parameters for the TCs.

Test cavities control

Control light is picked off from the heterodyne local oscillator (Het LO) at frequency $\omega_0 + \Lambda/2$ and frequency-shifted by $f_{\text{AOM}} = 80$ MHz using an AOM. PDH sidebands at 10 MHz are then generated with a resonant EOM. This control beam is injected into the test cavity as a p -polarized, counter-propagating mode relative to the s -polarized squeezing beam, thereby avoiding interference with the squeezed field. Quarter-wave plates (QWP) and polarizing beam splitters (PBS) are used to inject and extract the control beam. Because the test cavity is a triangular cavity, the resonance frequencies for s - and p -polarizations are separated by approximately $\text{FSR}_{\text{TC}}/2 \simeq 505$ MHz. Since the frequency separation between the Victor (ω_0) and Alice ($\omega_0 + \Lambda$) modes and the heterodyne local

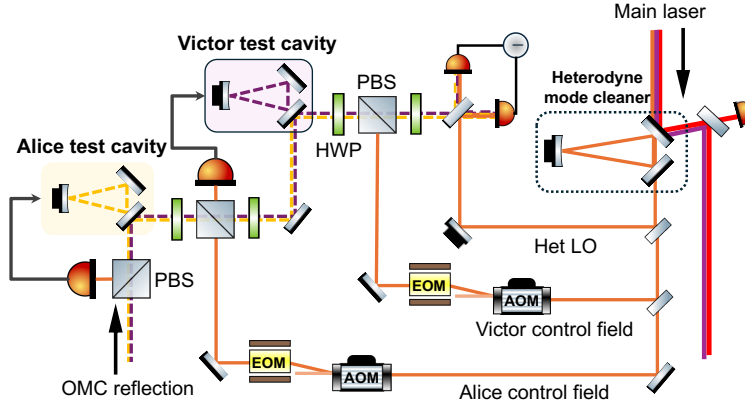


Figure 6.14: Schematic of the TC control system.

oscillator ($\omega_0 + \Lambda/2$) is $\pm \Lambda/2$, the p-polarized control field can be brought into simultaneous resonance with the s-polarized Victor and Alice frequency modes by choosing the AOM frequency such that

$$\frac{\text{FSR}_{\text{TC}}}{2} = \frac{\Lambda}{2} + f_{\text{AOM}}.$$

In this configuration, tuning the AOM drive frequency f_{AOM} provides controlled detuning of the test cavity with respect to the s-polarized Victor and Alice modes while maintaining cavity lock using the p-polarized control field. Figure 6.15 shows the transmitted p-polarized signal of the test cavity during cavity length control. From 0 to 11.5 s, the cavity length is scanned, during which resonance peaks are observed. After 11.5 s, the error signal is fed back to the actuator, and the cavity is stably locked near the resonance peak.

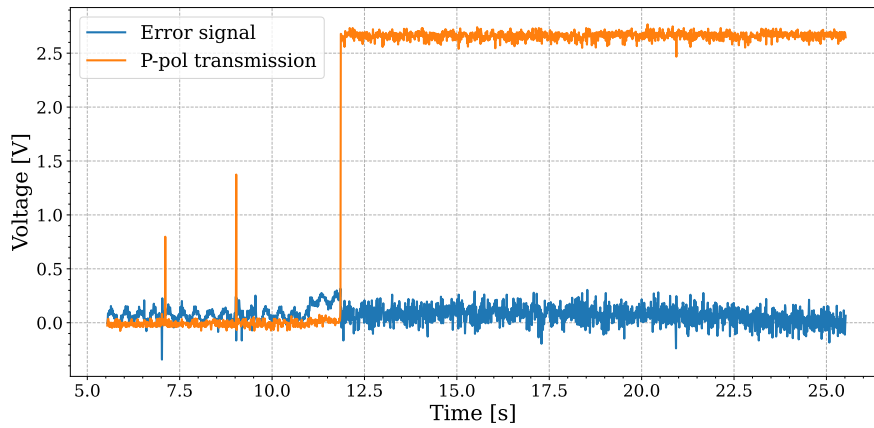


Figure 6.15: Transmitted p-polarized signal of the TC under length control.

6.3.5 Squeezing measurement

For the measurement of squeezing, a homodyne detector was implemented in the experimental setup. The detector was optimized by performing DC balance adjustment and

fringe visibility optimization. Subsequently, the squeezing arch was measured at a center frequency of 3.5 MHz.

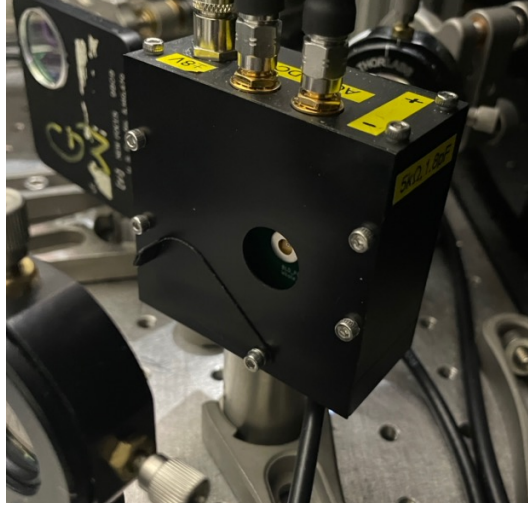


Figure 6.16: Image of the installed homodyne detector. The detector provides DC and AC outputs and allows squeezing measurement up to 20 MHz.

DC balance

DC balance is a procedure to balance the optical power incident on the two photodiodes of the homodyne detector, thereby suppressing classical common-mode noise contained in the input light. The DC balance is achieved by injecting only the LO beam and adjusting the beam-splitter angle until the DC photocurrents of the two photodiodes are equal, resulting in a zero differential DC output. The shot-noise and dark-noise spectra after DC balancing are shown in Fig. 6.17. The clearance between them is approximately 17 dB at 3.5 MHz with the local oscillator power of 7 mW at the beam splitter.

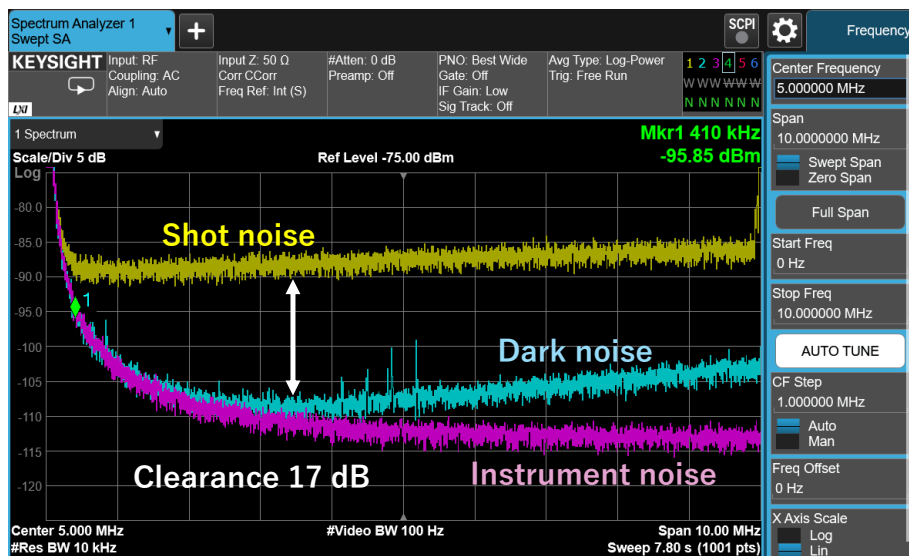


Figure 6.17: Shot-noise and dark-noise spectra of the homodyne detector after DC balancing.

Visibility measurement

Visibility measurement is an effective method for optimizing the overlap between the squeezing field and the local oscillator in terms of spatial mode matching, polarization, and alignment. By scanning the phase of the local oscillator with a PZT-mounted mirror, interference fringes are observed when the squeezing field and the local oscillator with equal optical power are injected into a photodetector. Maximizing the fringe visibility improves the overlap between the squeezing field and the local oscillator, leading to higher squeezing detection efficiency. The visibility $\mathcal{V}$ is defined as

$$\mathcal{V} = \frac{V_{\max} - V_{\min}}{V_{\max} + V_{\min} - 2V_{\text{off}}} \quad (6.3.12)$$

where $V_{\max}$ and $V_{\min}$ are the maximum and minimum voltages of the interference fringe, respectively. V_{off} is the offset voltage measured when all optical beams are blocked. Figure 6.18 shows the measured interference fringe. The red and blue dashed lines indicate the maximum and minimum voltages, respectively. The visibility is measured to be $96 \pm 1\%$.

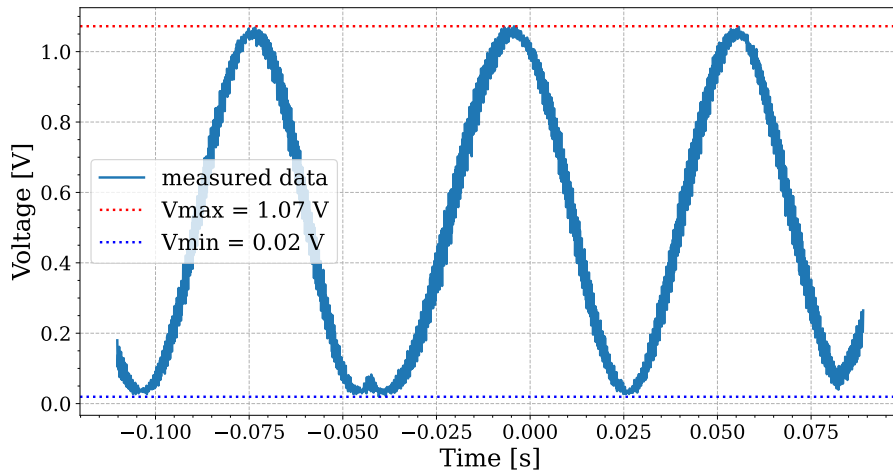


Figure 6.18: Interference fringe between the LO and seed light measured as the LO phase is scanned

The contribution of the visibility to the detection efficiency is given by

$$\eta_{\text{vis}} = \mathcal{V}^2 \simeq 0.92. \quad (6.3.13)$$

Squeezing arch measurement

We measured the single-mode squeezing arch using the Victor frequency mode. As a result, 1.4 dB of squeezing was observed at 3.5 MHz, as shown in Fig. 6.19. The squeezing and anti-squeezing levels were obtained from a zero-span homodyne measurement while scanning the LO phase. The measurement was performed with a local oscillator power of 7 mW and a shot-noise clearance of 20 dB.

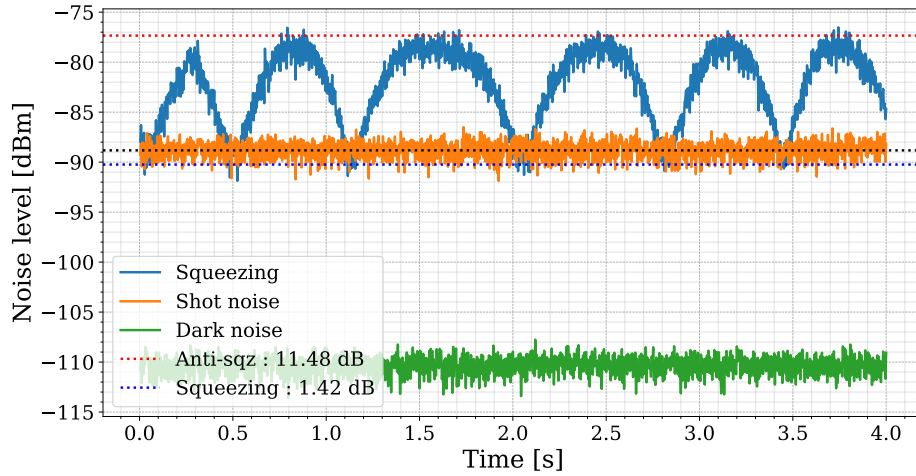


Figure 6.19: Squeezing arch measurement at a center frequency of 3.5 MHz with a sweep time of 4 s. The resolution bandwidth (RBW) and video bandwidth (VBW) were set to 10 kHz and 100 Hz, respectively.

The optical losses contributing to the squeezing measurement are summarized in Table 6.13. Compared with the squeezing level expected from this total efficiency, the experimentally observed squeezing is smaller than expected from the individually estimated efficiencies. This discrepancy is likely due to additional non-ideal effects, such as phase noise and mode mismatch, which are not fully captured by the simple loss model and effectively degrade the observed squeezing.

Parameter	Efficiency
OPO escape efficiency η_{eqc}	$98 \pm 1\%$
OMC transmission T_{OMC}	$84\% \pm 1\%$
Homodyne visibility η_{vis}	$92 \pm 1\%$
Homodyne quantum efficiency η_{QE}	$> 95\%$
Total efficiency η_{b}	$72 \pm 1\%$

Table 6.13: Efficiencies for each optical component.

6.4 QT Model result

In this section, we evaluate the noise spectrum achievable with quantum teleportation (QT) squeezing based on experimentally measured parameters. As discussed in Sec. 5, the quadratures of Alice and Bob generated by the OPO are described by Eq. (5.3.6), while the Victor quadratures are given by Eq. (5.3.11). In the following model calculation, individual loss contributions (e.g., OMC transmission, mode matching, and detector quantum efficiency) are incorporated into the total detection efficiency for each mode. Setting the homodyne angle to $\zeta = \pi/2$ and introducing the total detection efficiency of the homodyne detector η_B , which includes visibility and quantum efficiency, the measured quadrature becomes

$$\hat{B}_{\text{out}} = \sqrt{\eta_B} \hat{b}_2 + \sqrt{(1 - \eta_B)} \delta \hat{X}_2^{\text{vac}} \quad (6.4.1)$$

Alice and Victor are reflected by the OMC and subsequently experience quadrature rotations in the detuned test cavities. Since the reflectivities for Alice and Victor satisfy $R_{a,v} \simeq 1$, the quadratures after reflection from the test cavities are given by

$$\begin{aligned}\hat{\mathbf{A}} &= \begin{bmatrix} T_{11}(\Delta_a) & -T_{12}(\Delta_a) \\ T_{21}(\Delta_a) & T_{22}(\Delta_a) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \end{bmatrix} + \begin{bmatrix} T_{11}^l(\Delta_a) & -T_{12}^l(\Delta_a) \\ T_{21}^l(\Delta_a) & T_{22}^l(\Delta_a) \end{bmatrix} \begin{bmatrix} \delta\hat{X}_1^{\text{vac}} \\ \delta\hat{X}_2^{\text{vac}} \end{bmatrix} \\ \hat{\mathbf{V}} &= \begin{bmatrix} T_{11}(\Delta_v) & -T_{12}(\Delta_v) \\ T_{21}(\Delta_v) & T_{22}(\Delta_v) \end{bmatrix} \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix} + \begin{bmatrix} T_{11}^l(\Delta_v) & -T_{12}^l(\Delta_v) \\ T_{21}^l(\Delta_v) & T_{22}^l(\Delta_v) \end{bmatrix} \begin{bmatrix} \delta\hat{X}_1^{\text{vac}} \\ \delta\hat{X}_2^{\text{vac}} \end{bmatrix}\end{aligned}\quad (6.4.2)$$

where matrix elements are expressed as functions of the cavity detuning Δ

$$T_{11}(\Delta) = T_{22}(\Delta) = -1 + \frac{2\eta_{\text{esc}}(1 + i\omega/\kappa)}{(1 + i\omega/\kappa)^2 + (\Delta/\kappa)^2}, \quad T_{12}(\Delta) = T_{21}(\Delta) = \frac{2\eta_{\text{esc}}\Delta/\kappa}{(1 + i\omega/\kappa)^2 + (\Delta/\kappa)^2} \quad (6.4.3)$$

$$T_{11}^l(\Delta) = T_{22}^l(\Delta) = \frac{2\sqrt{\eta_{\text{esc}}(1 - \eta_{\text{esc}})}(1 + i\omega/\kappa)}{(1 + i\omega/\kappa)^2 + (\Delta/\kappa)^2}, \quad T_{12}^l(\Delta) = T_{21}^l(\Delta) = \frac{2\sqrt{\eta_{\text{esc}}(1 - \eta_{\text{esc}})}\Delta/\kappa}{(1 + i\omega/\kappa)^2 + (\Delta/\kappa)^2}. \quad (6.4.4)$$

For the joint measurement, we define $\hat{\mathbf{A}} = [\hat{V}_1, \hat{V}_2, \hat{A}_1, \hat{A}_2]^T$. With the heterodyne angle set to $\xi = \pi/2$, the heterodyne measurement yields the quadratures given in by

$$\begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\eta_V} \hat{V}_1 - \sqrt{\eta_A} \hat{A}_1 \\ \sqrt{\eta_V} \hat{V}_2 + \sqrt{\eta_A} \hat{A}_2 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{1 - \eta_V} \delta\hat{X}_1^v - \sqrt{1 - \eta_A} \delta\hat{X}_1^a \\ \sqrt{1 - \eta_V} \delta\hat{X}_2^v + \sqrt{1 - \eta_A} \delta\hat{X}_2^a \end{bmatrix}. \quad (6.4.5)$$

Using these measurement outcomes, we evaluate the combined variance

$$V(\hat{B}_{\text{out}} - g_1\alpha_1 - g_2\alpha_2). \quad (6.4.6)$$

Figure 6.20 shows the resulting variance obtained by applying the optimal filter using experimentally measured parameters together with assumed detection efficiencies listed in Table 6.14. The total detection efficiencies for the Alice and Victor modes were chosen based on the performance of the heterodyne detector reported in Ref. [48]. The figure shows that squeezing of up to approximately 0.5 dB below the shot-noise limit is achievable in a specific frequency band. Furthermore, the frequency-dependent behavior observed in both the QT squeezing and heterodyne variances originates from the quadrature rotation introduced by the detuned test cavities. Although there is a discrepancy between the measured FWHM of the test cavity and its design value, the frequency dependence observed in the variance remains within the squeezing band, and therefore does not pose a significant issue for the proof-of-principle experiment. Several key improvements remain in the experimental system introduced in this study. The first is improving the visibility of the homodyne detector. As discussed above, the visibility strongly depends on the polarization state, spatial mode matching, and alignment between the squeezing field and the local oscillator. By optimizing each of these factors and increasing the visibility to approximately 98 %, an improvement of about 4 % in the overall detection efficiency is expected, since the detection

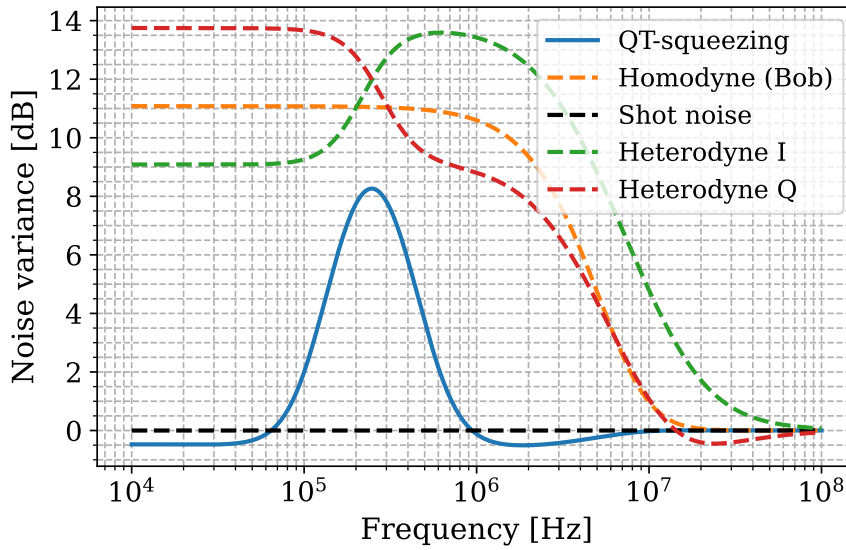


Figure 6.20: QT squeezing noise variance calculated using experimentally measured parameters and realistic detection efficiencies.

efficiency scales with the square of the visibility.

The second is improving the transmission of the OMC. By optimizing the mode matching to the OMC and fine-tuning the incidence angle, there remains room for improvement beyond the current transmission of approximately 84 %.

In addition, the detection efficiencies of Alice and Victor are particularly critical. As shown in Table 6.7, the visibility efficiency and quantum efficiency of the heterodyne detector are 97 % and 90 %, respectively. Under these conditions, achieving a total detection efficiency for Alice and Victor, $\eta_{A,V}$, exceeding 75 % in the model requires that the combined optical losses, including losses inside the test cavity, mode-matching losses, and propagation losses, be suppressed to better than 94 %. This requirement serves as an important benchmark for future performance improvements of the experimental system.

Parameter	Value
OPO escape efficiency η_{esc}	98 %
Filter gains g_1, g_2	$g_1 = g_2 = 0.95$
Nonlinear gain x	0.74
Half-linewidth of the test cavities κ_{TC}	0.17 MHz
Alice detuning Δ_a	0.17 MHz
Victor detuning Δ_v	0.17 MHz
Total Bob detection efficiency η_B	72 %
Total Alice detection efficiency η_A	75 %
Total Victor detection efficiency η_V	75 %

Table 6.14: Parameters used in the QT squeezing model calculation.

Chapter 7

Conclusion

7.1 Summary of the experiment

In this study, we constructed an experimental setup and characterized its individual components in order to perform a proof-of-principle demonstration of frequency-dependent squeezing based on quantum teleportation. In particular, we focused on the introduction and optimization of the key elements required for this scheme: the output mode cleaner (OMC), the test cavities, and the homodyne detector.

Stable length control of the OMC was successfully achieved, resulting in a transmission efficiency of 84 %. This enabled spatial separation of quantum states at different frequencies, namely the Bob, Alice, and Victor modes, thereby establishing an essential experimental foundation for the implementation of the QT squeezing scheme.

For the test cavities, length control was realized using a heterodyne local oscillator. Parameter measurements revealed that the reflectivities of the mirrors used in the experiment were higher than their design values, leading to a higher finesse and a narrower linewidth than initially intended. The cavity linewidth is a crucial parameter, as it determines both the difficulty of cavity control and the frequency response of the quadrature rotation induced by the cavity. Consequently, this linewidth constitutes an important constraint for future QT squeezing experiments.

In addition, the homodyne detector was optimized through DC balance and visibility measurements. As a result, a single-mode squeezing arch was successfully measured, yielding 11.48 dB of anti-squeezing and 1.42 dB of squeezing. This observed squeezing level is approximately 2 dB lower than that expected from the independently measured optical losses, suggesting the presence of unaccounted loss mechanisms and the influence of phase noise.

Finally, using experimentally measured parameters, we calculated a theoretical model for the noise variance achievable with QT squeezing. The results indicate that, with further improvements to the experimental system, an experimental demonstration of frequency-dependent squeezing based on the QT scheme is well within reach.

7.2 Future Work

The final experimental demonstration of the QT scheme is expected to proceed as follows. First, quantum entanglement and squeezed vacuum fields are generated in the OPO system. These states are then spatially separated by the OMC. Among the three quantum

states, Alice and Victor are sent through detuned test cavities, where frequency-dependent quadrature rotation is introduced. Alice and Victor are measured using heterodyne detection, while Bob is measured using homodyne detection. By applying appropriate post-processing to the measurement results, a variance spectrum such as that shown in Fig. 6.20 can be obtained, thereby completing the proof-of-principle experiment. Based on the results obtained in this study, several challenges remain to be addressed in order to achieve a full experimental demonstration of frequency-dependent squeezing using quantum teleportation.

- **Reduction of optical losses**

Achieving higher squeezing levels requires minimizing optical losses throughout the entire experimental setup. In particular, the transmission efficiency of the OMC can likely be further improved through optimized mode matching and fine adjustment of the input and output mirror alignment.

- **Control performance evaluation via transfer function measurements**

The stability and performance of the control systems should be quantitatively evaluated through transfer function measurements. This will clarify the achievable control bandwidth and phase margin, enabling further optimization of the control loops for long-term stable operation.

- **Detuning control of the test cavities**

In this study, the test cavity length was controlled using p-polarized light. Future work will extend this approach to precisely control the detuning experienced by the s-polarized squeezed light, allowing experimental verification of frequency-dependent quadrature rotation.

- **Implementation of a heterodyne detector**

Completion of the quantum teleportation protocol requires the implementation of a Bell measurement using heterodyne detection. The construction of the heterodyne detector and characterization of its noise performance therefore constitute important next steps.

- **Generation of quantum entanglement**

Quantum entanglement is a key resource for quantum teleportation. To verify the generation of entanglement, we plan to measure the combined variance using two homodyne detectors.

- **Control of the squeezing angle**

The squeezing measurements in this study were performed by scanning the local oscillator phase. For further technical validation, it will be necessary to actively control the squeezing angle and perform spectral measurements at fixed quadrature angles.

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